CHAPTER 2

ANALYSIS OF DETERMINATE STRUCTURES

This chapter focuses primarily on analysis of statically determinate structures. The key objective of the analysis is to determine unknown static quantities such as support reactions and internal forces resulting from external applied loads. As already discussed in the previous chapter, statically determinate structures belong to a special class of structures that reactions at all supports and the internal forces at any location of the structure can completely be determined from equilibrium equations. In following sections, we first emphasize the definition of unknown static quantities (i.e. support reactions and internal forces) and a notion of applied loads, and then discuss tools essential for performing static analysis of statically determinate structures, e.g. equilibrium equations, method of structure partitioning, and free body diagram (FBD). Next, we demonstrate applications of equilibrium equations to determine support reactions of externally, statically determinate structures. Finally, analysis for the internal forces for certain classes of structures such as trusses, beams and rigid frames are presented. In addition, for the case of beams and rigid frames, the sketch of their qualitative elastic curve (or deformed shape) is also discussed.

2.1 Static Quantities

Static quantities are quantities associated with forces, intensity of forces (e.g. pressure, traction, and stress), or resultants of forces (e.g. moment and torque). Three basic static quantities of primary interest in the analysis of structures are applied loads, support reactions, and internal forces as shown in Figure 2.1.

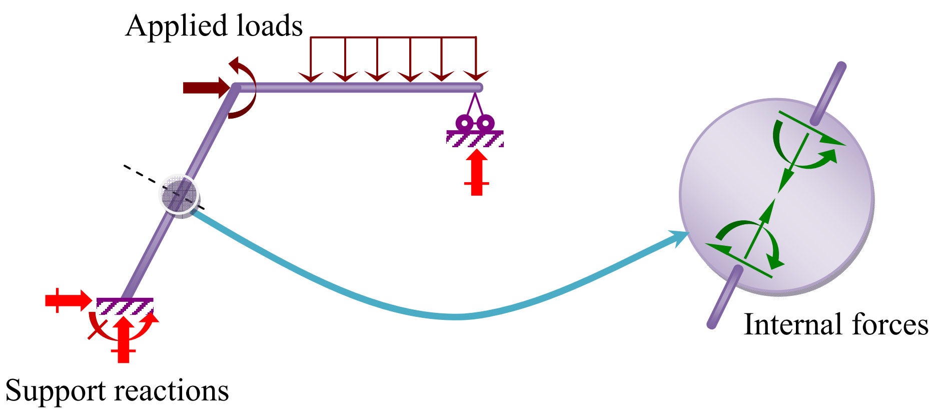


Figure 2.1: Schematic indicating applied loads, support reactions and internal forces

Applied loads represent *prescribed* forces, intensity of forces, resultants of forces, or in combination that are exerted to the structure by surrounding environments. It is emphasized that applied loads are known a priori (from idealization of excitations) before the analysis procedure is carried out.

Support reactions represent *unknown* forces, intensity of forces, resultants of forces, or in combination that are exerted to the structure by (idealized) supports to maintain its equilibrium and stability under the action of applied loads or other excitations. The support reactions are unknown a priori in nature and can only be obtained from static analysis.

Internal forces represent intensity of forces (e.g. stress) or resultant forces (e.g. axial force, shear force, bending moment, torque) induced at a material point or over a particular section of the structure under the action of applied loads or other excitations. Similar to the support reactions, the internal forces are unknown a priori and can only be obtained from static analysis. The choice of the internal forces used to characterize the behavior of any structure depends primarily on the type of structures and the nature of excitations. This will be discussed further below.

2.2 Tools for Static Analysis

Two key methodological components essential for determining support reactions and internal forces at any location of a structure are static equilibrium equations and the method of structure partitioning. The first component is utilized generally to construct a necessary and sufficient set of equations to solve for unknown support reactions and internal forces, while the latter component accommodates the construction of equilibrium equations over certain portions of the structure in addition to those associated with the entire structure in order to supply adequate number of equations.

2.2.1 Static equilibrium

Equilibrium condition of a body is a statement of conservation of linear momentum and angular momentum of that body. More precisely, the body is in equilibrium if and only if the linear momentum and angular momentum are conserved for any part of the body (the entire body can also be considered as a part of that body). For a two dimensional body occupied a region on the X-Y plane as shown schematically in Figure 2.2, equilibrium of this body implies that all forces and moments applied to the body must satisfy the following three equations:

$$\Sigma F_X = 0 \tag{2.1}$$

$$\Sigma F_Y = 0 \tag{2.2}$$

$$\Sigma M_{AZ} = 0 \tag{2.3}$$

where A is an *arbitrary* point used for computing the moment about the Z-axis and {X, Y, Z; O} denotes a reference Cartesian coordinate system. In particular, equations (2.1) and (2.2) indicate that the sum of components of all forces in X-direction and in Y-direction must vanish while the last equation (2.3) requires that the sum of moments about a point A in Z-direction must also vanish. As already pointed out in chapter 1, equilibrium condition of a two-dimensional body subjected to a system of general forces and moments provides exactly three independent equations (for a body subjected to special systems of applied loads, the number of independent equations can be less than three; the reader is suggested to consult section 1.7 in chapter 1 for extensive discussion).

It is important to emphasize that a set of three independent equations resulting from the equilibrium condition of a two-dimensional body is not unique. This non-uniqueness is due primarily to that the point for taking moment (i.e. point A) can be chosen arbitrarily (see discussion in section 1.7 in Chapter 1). Here, we present four different but equivalent sets of three equilibrium equations that can be employed in static analysis of two-dimensional structures.

2.2.1.1 Set 1

A set consists of the equilibrium of forces in X-direction (2.1), the equilibrium of forces in Y-direction (2.2), and the equilibrium of moments about an arbitrarily selected point A (2.3).

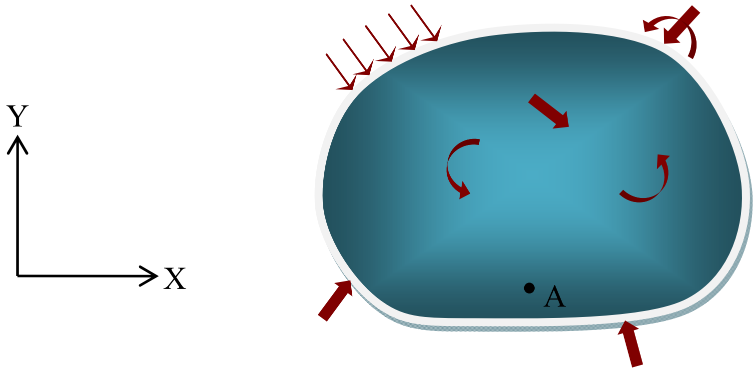


Figure 2.2: Schematic of two-dimensional body subjected to applied loads

2.2.1.2 Set 2

A set consists of the equilibrium of forces in X-direction (2.1), the equilibrium of moments about an arbitrarily selected point A (2.3), and the equilibrium of moments about another arbitrary selected point B, i.e.

$$\Sigma M_{BZ} = 0 \tag{2.4}$$

The only constraint placed on the choice of points A and B to render (2.1), (2.3) and (2.4) all independent is that the straight line connecting A and B must not parallel to the Y-axis.

2.2.1.3 Set 3

A set consists of the equilibrium of forces in Y-direction (2.2), the equilibrium of moments about an arbitrarily selected point A (2.3), and the equilibrium of moments about another arbitrary selected point B (2.4). Again, the only constraint placed on the choice of points A and B to render (2.2), (2.3) and (2.4) all independent is that the straight line connecting A and B must not parallel to the X-axis.

2.2.1.4 Set 4

A set consists of the equilibrium of moments about an arbitrarily selected point A (2.3), the equilibrium of moments about an arbitrary selected point B (2.4), and the equilibrium of moments about an arbitrary selected point C, i.e.

$$\Sigma M_{CZ} = 0 \tag{2.5}$$

The only constraint placed on the choice of points A, B and C to render (2.3), (2.4) and (2.5) all independent is that the A, B and C must not belong to the same straight line.

To prove the equivalence among the above four sets, let's assume that the body is subjected to a system of forces P_1, P_2, …, and P_N acting at points (X_1, Y_1), (X_2, Y_2), …, (X_N, Y_N), respectively, and a system of moments M_1, M_2, …, and M_K as shown in Figure 2.3. Let A, B, and C be three arbitrarily selected points with coordinates (X_A, Y_A), (X_B, Y_B), and (X_C, Y_C), respectively. The equilibrium of forces in the X-direction and Y-direction takes the form

$$\sum_{i=1}^{N} F_{iX} = 0 \tag{2.6}$$

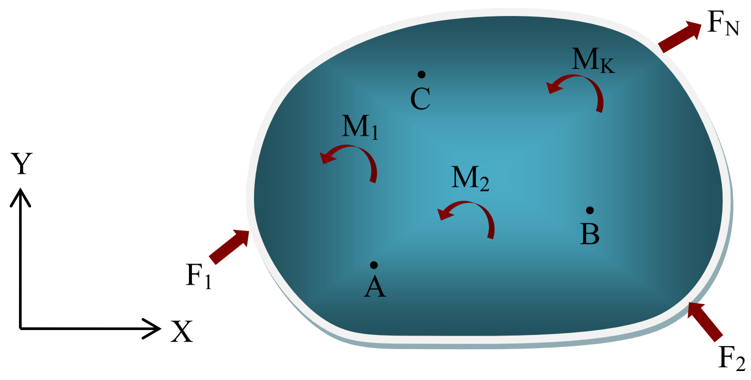


Figure 2.3: Schematic of two-dimensional body subjected to a system of forces and moments

$$\sum_{i=1}^{N} F_{iY} = 0 \tag{2.7}$$

where the subscripts "X" and "Y" indicate the X-component and Y-component of the force F_i, respectively. Similarly, equilibrium of moments about point A, B and C can readily be obtained as

$$\sum_{i=1}^{N} F_{iX}(Y_A - Y_i) + \sum_{i=1}^{N} F_{iY}(X_i - X_A) + \sum_{i=1}^{K} M_i = 0 \tag{2.8}$$

$$\sum_{i=1}^{N} F_{iX}(Y_B - Y_i) + \sum_{i=1}^{N} F_{iY}(X_i - X_B) + \sum_{i=1}^{K} M_i = 0 \tag{2.9}$$

$$\sum_{i=1}^{N} F_{iX}(Y_C - Y_i) + \sum_{i=1}^{N} F_{iY}(X_i - X_C) + \sum_{i=1}^{K} M_i = 0 \tag{2.10}$$

Upon simple manipulations, equation (2.9) can be expressed as

$$\sum_{i=1}^{N} F_{iX}(Y_A - Y_i) + \sum_{i=1}^{N} F_{iY}(X_i - X_A) + \sum_{i=1}^{K} M_i + (Y_B - Y_A)\sum_{i=1}^{N} F_{iX} + (X_A - X_B)\sum_{i=1}^{N} F_{iY} = 0 \tag{2.11}$$

Similarly, equation (2.10) can also be written as

$$\sum_{i=1}^{N} F_{iX}(Y_A - Y_i) + \sum_{i=1}^{N} F_{iY}(X_i - X_A) + \sum_{i=1}^{K} M_i + (Y_C - Y_A)\sum_{i=1}^{N} F_{iX} + (X_A - X_C)\sum_{i=1}^{N} F_{iY} = 0 \tag{2.12}$$

It is evident from (2.11) and (2.12) that equations (2.9) and (2.10) are not independent of the three equations (2.6), (2.7) and (2.8) but, in fact, they are the linear combinations of those three. Thus, the Set 1 is equivalent to the Set 2 provided that $X_A - X_B$ does not vanishes or, in the other word, a straight line connecting points A and B is not parallel to the Y-axis. It can readily be verified by employing equations (2.6) and (2.8) and then dividing the final result by $X_A - X_B$ that (2.11) can be reduced to equation (2.7). This similar argument can also be used to prove the equivalence between the Set 3 and the Set 1. Finally, the equivalence between the Set 4 and the Set 1 can be verified by first substituting (2.8) into (2.11) and (2.12) to obtain

$$(Y_B - Y_A)\sum_{i=1}^{N} F_{iX} + (X_A - X_B)\sum_{i=1}^{N} F_{iY} = 0 \tag{2.13}$$

$$(Y_C - Y_A)\sum_{i=1}^{N} F_{iX} + (X_A - X_C)\sum_{i=1}^{N} F_{iY} = 0 \tag{2.14}$$

Equations (2.13) and (2.14) are equivalent to equations (2.6) and (2.7) if and only if $(Y_B - Y_A)(X_A - X_C) - (Y_C - Y_A)(X_A - X_B) \neq 0$ or, equivalently, the three points A, B, C does not belong to the same straight line.

There is no strong evidence to support and decide the best choice from the four sets given above. In general, the choice is a matter of taste and preference of an individual and, sometimes, it is problem dependent. The most reasonable choice is the one that allows all unknowns appearing in all three equations be solved in an easy manner as much as possible. In addition, after one of the four sets is already chosen, the order of three equations in the set to be employed and the choice of points used for taking moments are generally selected to eliminate the unknowns as many as possible in order to avoid solving a large system of linear equations. This strategy becomes more apparent in examples 2.1-2.3.

2.2.2 Method of structure partitioning

To determine the internal forces at any location of the structure or to compute some components of the support reactions for certain structures, the consideration of equilibrium of certain parts of the structure is required in addition to that of the entire structure. From the fact that the structure is in equilibrium if and only if any part of the structure is in equilibrium, any portion of the structure resulting from partitioning of the structure must be in equilibrium with applied loads acting to that portion (i.e. reactions at supports present in that portion and internal forces exerted to that portion by the rest of the structure). *Structure partitioning* is simply a process to decompose the structure into two or several parts by introducing a sufficient number of fictitious or imaginary cuts at certain locations of the structures. For instance, a rigid frame shown in Figure 2.4 is partitioned into two parts by a fictitious cut at point B. One crucial function of the fictitious cut is to allow us to access and see the internal forces at the location of the cut. As can be observed from free body diagrams of the two parts in Figure 2.4, the internal forces $\{F_B, V_B, M_B\}$ at point B appear in both FBDs.

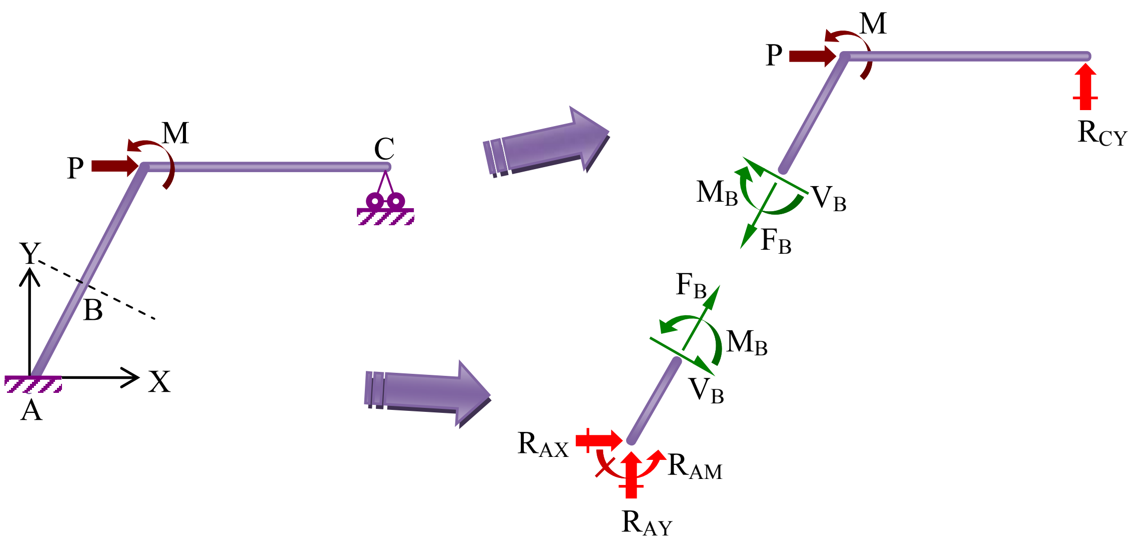


Figure 2.4: Schematic of the entire structure and two parts resulting from partitioning at point B

Since each portion of the structure resulting from partitioning must be in equilibrium, several equilibrium equations are therefore provided in addition to the equilibrium equations set up on the entire structure. However, the number of unknowns associated with the internal forces appearing at the cuts also increases at the same time. Hence, locations of the cuts and the number of the cuts are important and must properly be chosen in order to ensure that the number of all unknowns does not exceed the number of available equilibrium equations. In addition, the imaginary cut must be made at the point where the internal forces are of interest. For instance, the cut must be made at point B of the rigid frame in Figure 2.4 if the internal forces $\{F_B, V_B, M_B\}$ are to be determined.

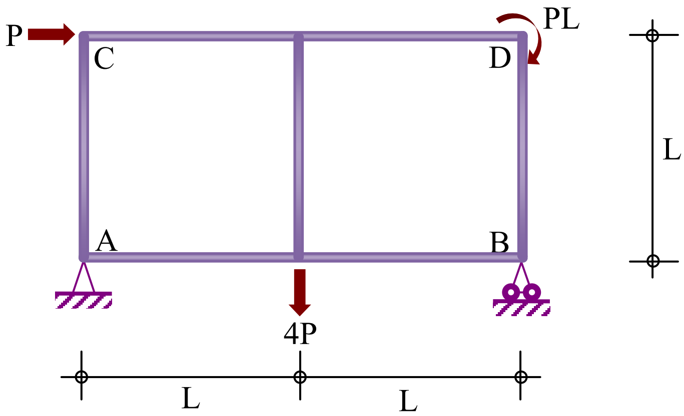
2.3 Determination of Support Reactions

This section demonstrates the application of static equilibrium to compute all support reactions of externally, statically determinate structures (note that statically determinate structures are also contained in this class of structures). Recalling the definition provided in subsection 1.8.2 of chapter 1, all support reactions of externally, statically determinate structures can completely be obtained by solving static equilibrium equations. A brief summary of guidelines for determining support reactions is given below:

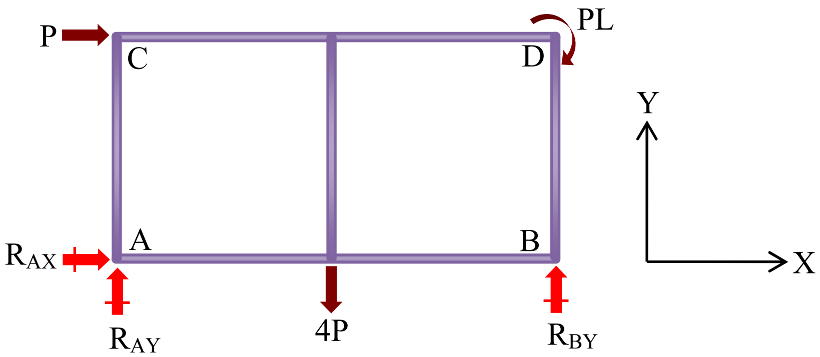
- Identify type of all supports
- Determine the number of unknown support reactions (r_a)
- Determine the number of independent equilibrium equations that can be set up for the entire structure (n_{et})
- Determine the number of additional static conditions that can be set up without introducing new static unknowns (n_{cr}); consult subsection 1.9.2 of chapter 1 for extensive discussion
- If $r_a > n_{et} + n_{cr}$, the structure is externally statically indeterminate and support reactions cannot completely be obtained by using only equilibrium equations. If $r_a = n_{et} + n_{cr}$, the structure is externally statically determinate and support reactions can be determined as described below. It is worth noting that if a given structure is known to be statically determinate (i.e. DI = 0), it automatically implies that $r_a = n_{et} + n_{cr}$.
- If $r_a = n_{et}$, all support reactions can be obtained from equilibrium of the entire structures and the following steps are suggested: (i) sketch a free body diagram (FBD) of the entire structure, (ii) write down all independent equilibrium equations, and (iii) solve for unknown support reactions
- If $r_a > n_{et}$, all support reactions cannot be obtained from equilibrium of the entire structures alone and the subsequent steps are as follows: (i) introduce suitable fictitious cuts; in general, fictitious cuts are made at the internal releases present within the structure (e.g. hinges, shear releases, axial releases), (ii) ensure that the number of fictitious cuts is sufficient to supply adequate number of equations to solve for all unknowns (i.e. all support reactions and extra unknowns associated with the internal forces induced at the cuts), (iii) sketch a free body diagram of the entire structure and free body diagrams of parts resulting from the cuts, (iv) write down all independent equilibrium equations for the entire structure and for parts resulting from the cuts, and (v) solve for unknown support reactions.

It is important to emphasize again that after the free body diagram(s) is sketched, a careful choice of reference points for computing moment and the order of equilibrium equations employed can significantly reduce the computational effort associated with solving linear equations. This becomes apparent in following examples.

<u>Example 2.1</u> Determine all support reactions of a rigid frame shown below



<u>Solution</u> The given structure is constrained at points A and B by a pinned support and a roller support, respectively; thus, the total number of support reactions is $r_a = 2 + 1 = 3$. The number of independent equilibrium equations for a two-dimensional rigid frame is $n_{et} = 3$. Since $r_a = n_{et}$, the structure is externally, statically determinate and all support reactions can be obtained by considering equilibrium of the entire structure. FBD of the entire structure is given below where $\{R_{AX}, R_{AY}\}$ and R_{BY} are unknown support reactions at point A and point B, respectively. To demonstrate the equivalence among four sets of equilibrium equations mentioned in section 2.2.1, we apply each set separately and then compare the final results.



<u>Option I</u>: Use equilibrium equations from the Set 1

$[\Sigma F_X = 0]$ $\rightarrow +$: $R_{AX} + P = 0$

$R_{AX} = -P$ Leftward

$[\Sigma M_A = 0]$ $\circlearrowleft +$: $(R_{BY})(2L) - PL - (P)(L) - (4P)(L) = 0$

$R_{BY} = 3P$ Upward

$[\Sigma F_Y = 0]$ $\uparrow +$: $R_{AY} + R_{BY} - 4P = 0$

$R_{AY} = P$ Upward

<u>Option II</u>: Use equilibrium equations from the Set 2

$[\Sigma F_X = 0]$ $\rightarrow +$: $R_{AX} + P = 0$

$R_{AX} = -P$ Leftward

$[\Sigma M_A = 0]$ ↺+ : $(R_{BY})(2L) - PL - (P)(L) - (4P)(L) = 0$

$R_{BY} = 3P$ Upward

$[\Sigma M_B = 0]$ ↺+ : $-(R_{AY})(2L) + (4P)(L) - (P)(L) - PL = 0$

$R_{AY} = P$ Upward

<u>Option III</u>: Use equilibrium equations from the Set 3

$[\Sigma M_A = 0]$ ↺+ : $(R_{BY})(2L) - PL - (P)(L) - (4P)(L) = 0$

$R_{BY} = 3P$ Upward

$[\Sigma F_Y = 0]$ ↑+ : $R_{AY} + R_{BY} - 4P = 0$

$R_{AY} = P$ Upward

$[\Sigma M_C = 0]$ ↺+ : $(R_{AX})(L) + (R_{BY})(2L) - (2P)(L) - PL = 0$

$R_{AX} = -P$ Leftward

<u>Option IV</u>: Use equilibrium equations from the Set 4

$[\Sigma M_A = 0]$ ↺+ : $(R_{BY})(2L) - PL - (P)(L) - (4P)(L) = 0$

$R_{BY} = 3P$ Upward

$[\Sigma M_B = 0]$ ↺+ : $-(R_{AY})(2L) + (4P)(L) - (P)(L) - PL = 0$

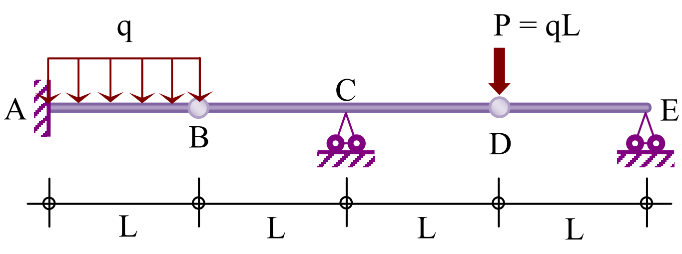
$R_{AY} = P$ Upward

$[\Sigma M_C = 0]$ ↺+ : $(R_{AX})(L) + (R_{BY})(2L) - (4P)(L) - PL = 0$

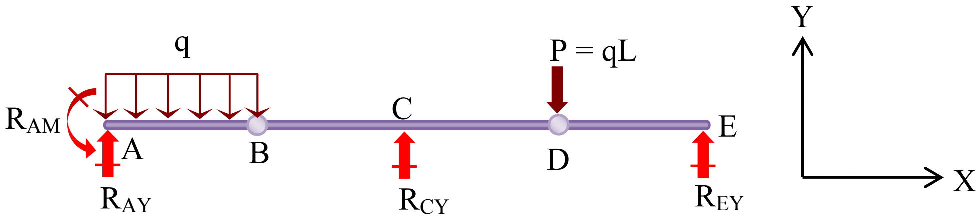
$R_{AX} = -P$ Leftward

It is evident that use of any set of equilibrium equations leads to the same results. While the order of equilibrium equations employed does not matter, those containing only one unknown are considered first. This allows the unknown be easily obtained without solving any system of linear equations.

<u>Example 2.2</u> Determine all support reactions of a beam shown below



<u>Solution</u> Since $r_a = 2 + 1 + 1 = 4$, $m = 2$, $n = 3$, $n_c = 2$, then $DI = 4 + 2(2) - 2(3) - 2 = 0$. Thus, the structure is statically determinate and all support reactions can be determined form static equilibrium. However, the number of independent equilibrium equations that can be set up for a beam is $n_{et} = 2 < r_a$; thus, the support reactions cannot be obtained by considering only equilibrium of the entire structure. To clarify this, let us sketch the FBD of the entire beam as shown below.



Equilibrium of the entire beam requires that

$[\Sigma M_A = 0]$ ↺+ : $R_{AM} + (R_{CY})(2L) + (R_{EY})(4L) – (qL)(L/2) – (qL)(3L) = 0$

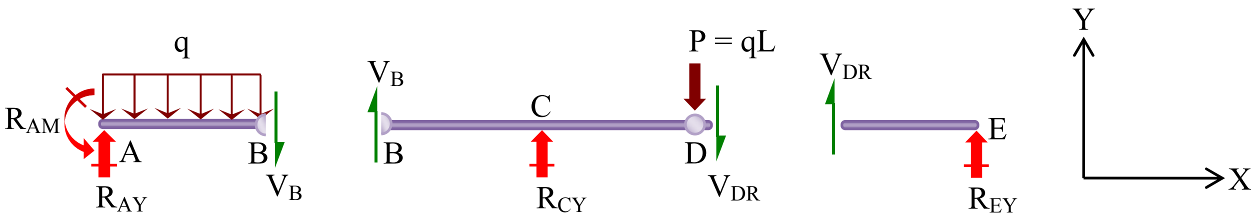
$R_{AM} + 2R_{CY}L + 4R_{EY}L = 7qL^2/2$

$[\Sigma F_Y = 0]$ ↑+ : $R_{AY} + R_{CY} + R_{EY} – qL – qL = 0$

$R_{AY} + R_{CY} + R_{EY} = 2qL$

It is emphasized that there are only two independent equilibrium equations and they are insufficient to be solved for four unknown support reactions R_{AM}, R_{AY}, R_{CY} and R_{EY}. To overcome this problem, two additional equations associated with the presence of two moment releases or hinges at points B and D, i.e. M = 0 at B and M = 0 at D, must be employed.

By introducing two cuts, one at point B and the other at point just to the right of point D called DR, the original structure is decomposed into three parts and the FBD of each part is shown below. Note that the cut is not made exactly at point D due to the application of a concentrated load at point D and we choose to avoid the question on how to distribute this concentrated load to the left and right parts of the point D. For this particular choice of the cut (cut at point DR), the concentrated load P = qL appears at the point DR of the FBD of the middle part.



The left portion contains three unknowns, i.e., $\{R_{AM}, R_{AY}, V_B\}$; the middle portion also contains three unknowns, i.e., $\{R_{CY}, V_B, V_{DR}\}$; and the right portion contains only two unknowns, i.e., $\{V_{DR}, R_{EY}\}$. The total number of unknowns (including all support reactions and the shear forces appearing at the cuts) now becomes six, i.e., $\{R_{AM}, R_{AY}, R_{CY}, R_{EY}, V_B, V_{DR}\}$. Two independent equilibrium equations can be set up for each individual portion and this leads to a set of six independent linear equations sufficient for determining all unknowns.

To avoid solving a system of six linear equations, we first consider the right portion in which the number of unknowns is equal to the number of independent equilibrium equations. By first applying equilibrium of moments about point E and then considering equilibrium of forces in Y-direction, we obtain

$[\Sigma M_E = 0]$ ↺+ : $V_{DR}L = 0$

$V_{DR} = 0$

$[\Sigma F_Y = 0]$ ↑+ : $R_{EY} + V_{DR} = 0$

$R_{EY} = 0$

Since the shear force V_{DR} is already known, the number of unknowns for the middle portion now reduces to two, i.e., $\{R_{CY}, V_B\}$, and they can be solved by considering equilibrium of this portion as follows:

$[\Sigma M_B = 0]$ ↺+ : $(R_{CY})(L) – (qL)(2L) – (V_{DR})(2L) = 0$

$R_{CY} = 2qL$ Upward

$[\Sigma F_Y = 0]$ ↑+ : $V_B + R_{CY} – qL – V_{DR} = 0$

$V_B = –qL$

Since the shear force V_B is already known, the number of unknowns for the left portion now reduces to two, i.e., $\{R_{AM}, R_{AY}\}$, and they can be solved by considering equilibrium of this portion as follows:

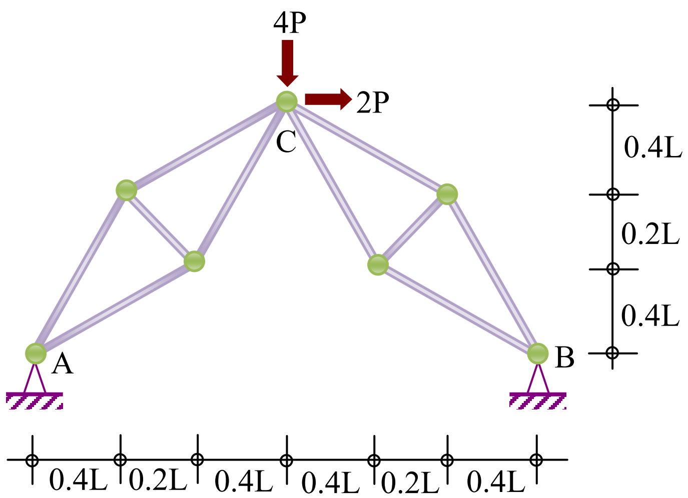
$[\Sigma M_A = 0]$ ↺+ : $R_{AM} – (qL)(L/2) – (V_B)(L) = 0$

$R_{AM} = –qL^2/2$

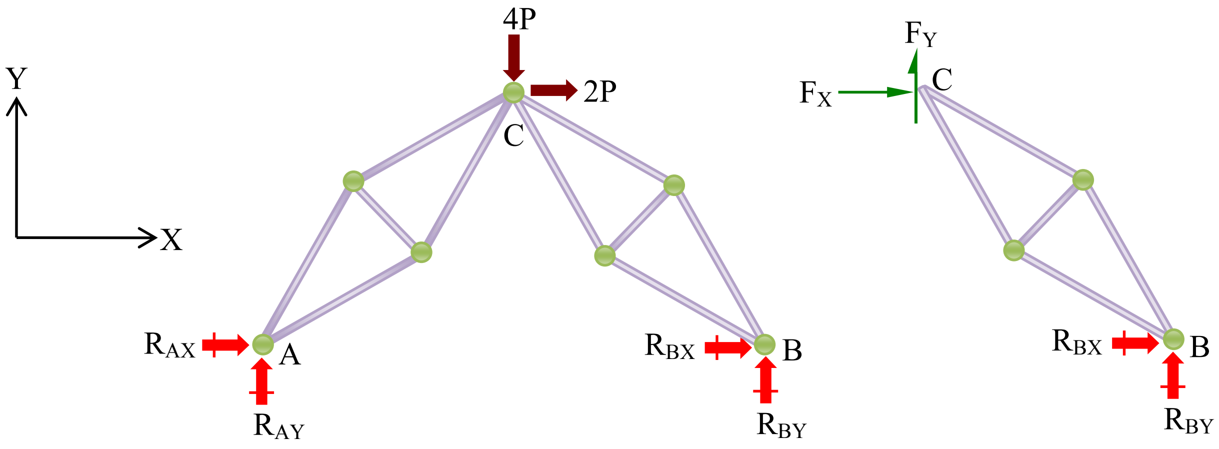
$[\Sigma F_Y = 0]$ ↑+ : $R_{AY} – qL – V_B = 0$

$R_{AY} = 0$

<u>Example 2.3</u> Determine all support reactions of a truss shown below



<u>Solution</u> Since $r_a = 2 + 2 = 4$, $m = 10$, $n = 7$, $n_c = 0$, then DI = 4 + 10(1) – 7(2) – 0 = 0. Thus, the structure is statically determinate and all support reactions can be determined from static equilibrium. However, the number of independent equilibrium equations that can be set up for a beam is n_{et} = 3 < r_a; thus, the support reactions cannot be obtained by considering only the equilibrium of the entire structure. Partitioning of the structure to construct additional equations is then required.



First, let us consider the equilibrium of the entire structure (its FBD is shown above). By applying equilibrium of moments about point A and B, the two support reactions {R_{AY}, R_{BY}} can readily be determined:

[$\Sigma M_A = 0$] ↺+ : $(R_{BY})(2L) - (2P)(L) - (4P)(L) = 0$

$R_{BY} = 3P$ Upward

[$\Sigma M_B = 0$] ↺+ : $-(R_{AY})(2L) - (2P)(L) + (4P)(L) = 0$

$R_{AY} = P$ Upward

The remaining equilibrium equation associated with equilibrium of forces in X-direction only provides a relation between the two unknowns {R_{AX}, R_{BX}}:

[$\Sigma F_A = 0$] →+ : $R_{AX} + R_{BX} + 2P = 0$

To obtain an additional equation, we introduce a cut at point just to the right of point C and then consider the right portion resulting from that cut; the FBD is shown in the figure below. Since point C is a pinned or hinge joint, only two new unknowns {F_X, F_Y} are introduced at the cut. In addition, the applied loads acting at point C do not appear in the FBD of the right portion due to that the cut is not made exactly at point C but at point just to the right of point C. By considering equilibrium of moments about point C and then using the known value of R_{BY}, we obtain

[$\Sigma M_C^+ = 0$] ↺+ : $(R_{BX})(L) + (R_{BY})(L) = 0$

$R_{BX} = -3P$ Leftward

Once R_{BX} is known, the reaction R_{AX} can readily be obtained from above equation, i.e., $R_{AX} = -2P - R_{BX} = P$ (Rightward).

2.4 Static Analysis of Truss

In this section, we focus our attention on the analysis of statically determinate trusses with the primary objective to determine all support reactions and internal forces due to external applied loads. The section starts with a brief introduction to characteristics of trusses and notations and terms used throughout. Next, we present two standard methods typically employed to determine the internal force of each member of the truss. Various examples are then presented to demonstrate the principle and procedure of each method.

2.4.1 Characteristics of truss

An idealized structure is termed a *truss* if and only if (i) all members are straight, (ii) all members are connected by pinned (or hinge or truss) joints, (iii) all applied loads are in terms of concentrated forces and they act only at joints, and (iv) all supports provides only constraints against translations. Examples of trusses are shown schematically in Figure 2.5. Note that concentrated moments are not allowed to be applied to any joint of the truss since they provide no rotational constraint.

From above definition, it can readily be verified that any member of the truss possesses only one component of the internal force (i.e. the *axial force*) and this axial force is *constant* throughout the member. In addition, the angle between any two members joining the same pinned joint is not preserved; the angle measured before and after application of applied loads is generally not identical. To demonstrate this feature, let us consider a truss shown in Figure 2.6. After subjected to applied loads, this truss is deformed to a new state and the configuration associated with this state is termed the deformed configuration (represented by a dash line). Clearly, angles between any two members before and after movement are not necessarily the same, e.g. $\theta_0 \neq \theta$ and $\beta_0 \neq \beta$.

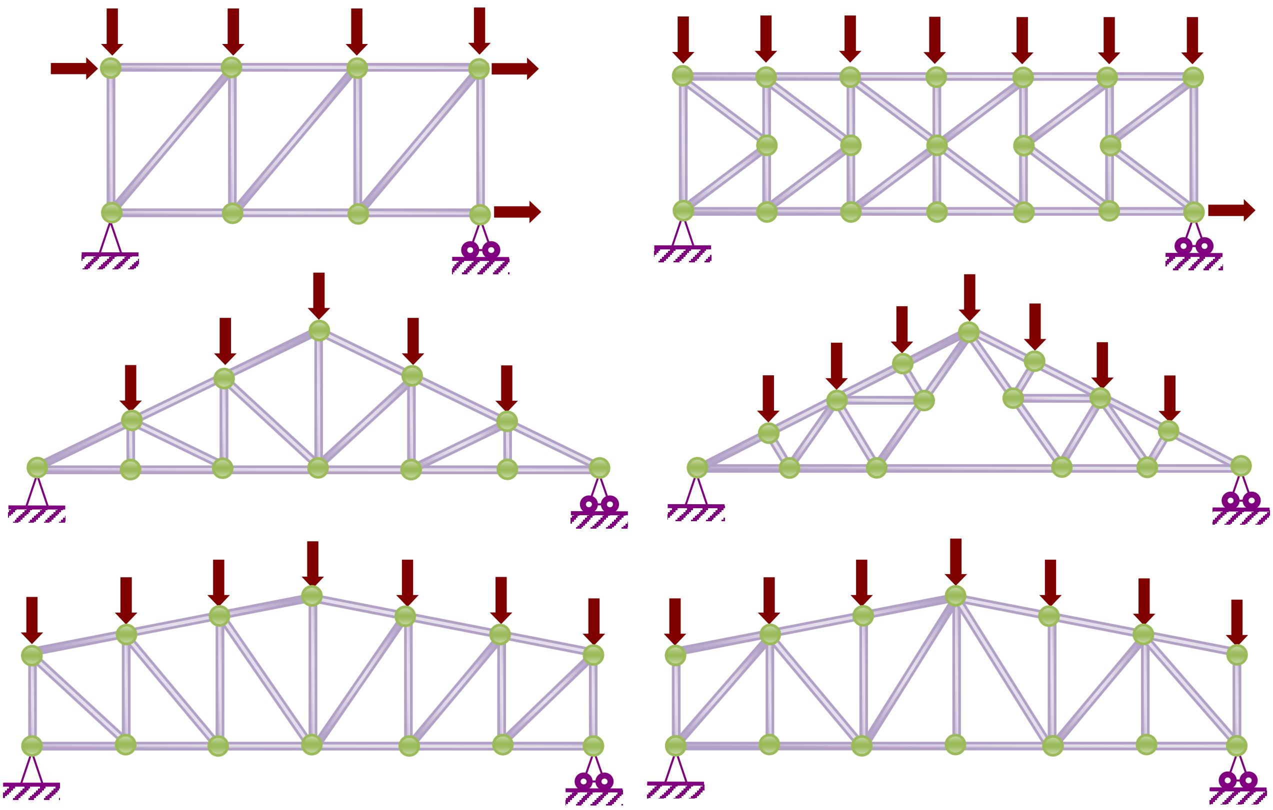


Figure 2.5: Example of truss structures

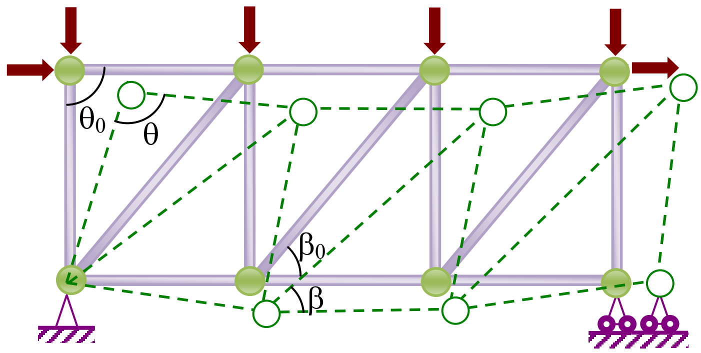


Figure 2.6: Schematic of undeformed and deformed configurations of a truss

2.4.2 Sign and convention

For support reactions of a given truss, there is no specific notation for their label. Typically, they are named based on the label of the joint where the support is located and their direction with respective to a reference coordinate system. For example (see Figure 2.7), support reactions induced at a pinned support located at a point A can be labeled R_{AX} and R_{AY} where the first one denotes the reaction at the point A in the X-direction and the last one denotes the reaction at the point A in the Y-direction, and, similarly, the support reaction induced at a roller support located at a point B can be labeled R_{BY}. Note that the upper case R is used only to distinguish the reaction from the other two static quantities, applied loads and internal forces. The sign convention of the support reaction is defined based on the reference coordinate system; specifically, it is positive if it directs along the positive coordinate direction otherwise it is negative. In the analysis for support reactions, it is typical to assume a priori that all components of support reactions are positive or direct along the coordinate directions (as shown in Figure 2.7) in order to prevent any confusion and error. The actual direction of all support reactions can be known once the analysis is complete. If the value of the support reaction obtained is positive, the assumed direction of such reaction is correct, and if its value is negative, the assumed direction of such reaction is wrong and must be reversed.

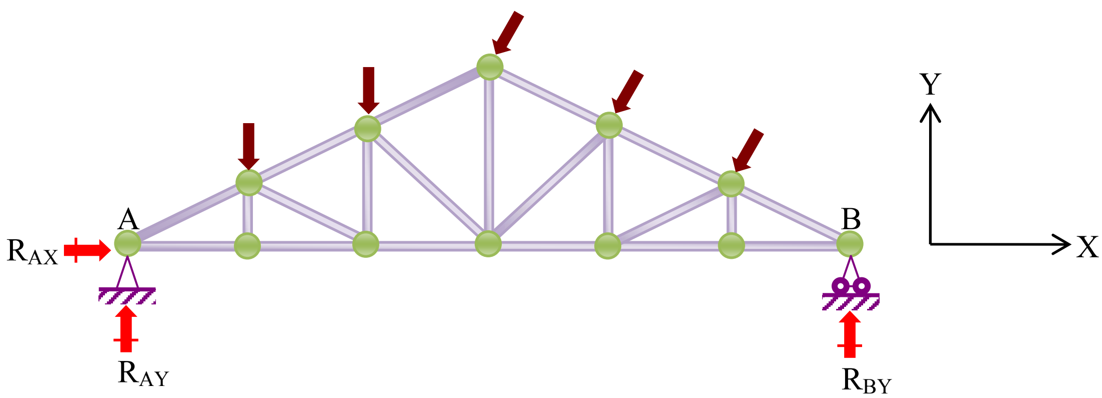


Figure 2.7: Labels of support reactions and their sign convention

For the internal force or member force of trusses, it is standard to follow notations and sign convention as described below:

- A member joining joint *i* and joint *j* is called "member *ij*" or equivalently "member *ji*".

- The internal force (or the axial force or the member force) of a member *ij* is denoted by F_{ij}. The internal force F_{ij} is positive if and only if the member *ij* is in tension and negative if and only if the member *ij* is in compression.

Figure 2.8 shows the schematic of the internal force at both ends of the member *ij* and at the joints *i* and *j* for a member in tension and a member in compression. It is worth pointing out that for a member in tension, the internal force directs outward from the joint in the FBD of that joint and directs outward from the member end in the FBD of that member and this observation is reverse if the member is in compression.

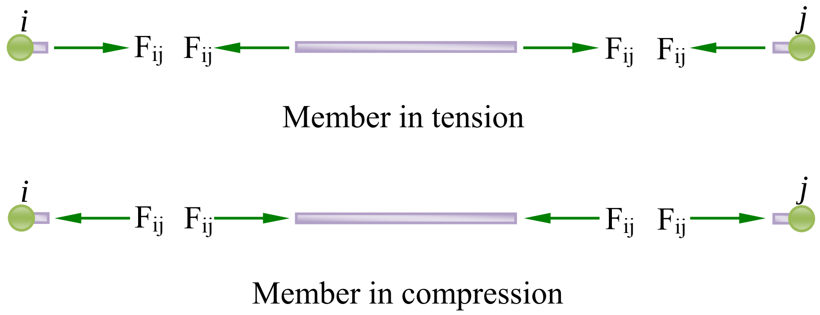


Figure 2.8: Schematic of the internal force in FBD of joints and member

2.4.3 Determination of support reactions

In the analysis of statically determinate trusses, all support reactions are generally computed first before the process in determining the internal or member forces starts. The procedure to obtain these quantities for truss structure follows exactly that given in the section 2.3. For some trusses such as those shown in Figure 2.5, the number of support reactions is equal to 3 and, hence, the consideration of equilibrium of the entire structure provides a sufficient number of equations to solve for those unknowns. For instance, the truss shown in Figure 2.7 has three unknown support reactions {R_{AX}, R_{AY}, R_{BY}} and they can readily be computed as follow:

- the reaction R_{BY} is obtained from equilibrium of moment about point A of the entire structure;
- the reaction R_{AY} is obtained from equilibrium of forces in Y-direction of the entire structure; and
- the reaction R_{AX} is obtained from equilibrium of forces in X-direction of the entire structure.

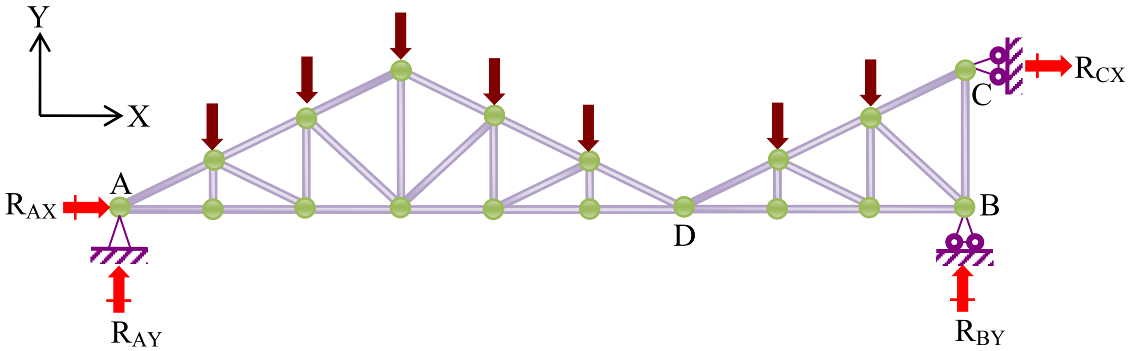


Figure 2.9: Schematic of a truss that a number of support reactions is more than three

For some trusses, the number of support reactions may exceed three, while they are still statically determinate (see for example a truss shown in Figure 2.9). In this case, it requires, in addition, the consideration of equilibrium of certain parts of the structure that results from fictitious cuts at proper locations such as a joint connecting between two sub-trusses (see for example a joint D of a truss shown in Figure 2.9). The need for introducing cuts is to supplement extra equations sufficient for, when combined with a set of equilibrium equations for the entire structure, solving all unknown reactions.

To clearly demonstrate the above argument, consider, for example, a truss shown in Figure 2.9. This structure is obviously statically determinate (i.e., $r_a = 4$, $n_m = 32(1) = 32$, $n_j = 18(2) = 36$, $n_c = 0 \rightarrow \text{DI} = 4 + 32 - 36 - 0 = 0$) and this therefore ensures that all support reactions can be obtained from static equilibrium. However, it is evident that all four support reactions $\{R_{AX}, R_{AY}, R_{BY}, R_{CX}\}$ cannot be obtained by considering only equilibrium of the entire structure (it yields only three independent equations). To overcome such degeneracy, we can introduce a cut at the point D and then separate the truss into two parts as shown in Figure 2.10. While we introduce two extra unknowns $\{F_X, F_Y\}$ at the cut, the total number of unknowns (4 + 2 = 6) is now equal to the number of equilibrium equations that can be set up for the two parts (3 + 3 = 6). To obtain all support reactions without solving a system of six linear equations, equilibrium of both the entire structure and its parts may be considered together as follow:

- the reaction R_{AY} is obtained from equilibrium of moments about the point D of the left part;
- the reaction R_{BY} is obtained from equilibrium of forces in the Y-direction of the entire structure;
- the reaction R_{CX} is obtained from equilibrium of moments about the point D of the right part; and
- the reaction R_{AX} is obtained from equilibrium of forces in the X-direction of the entire structure.

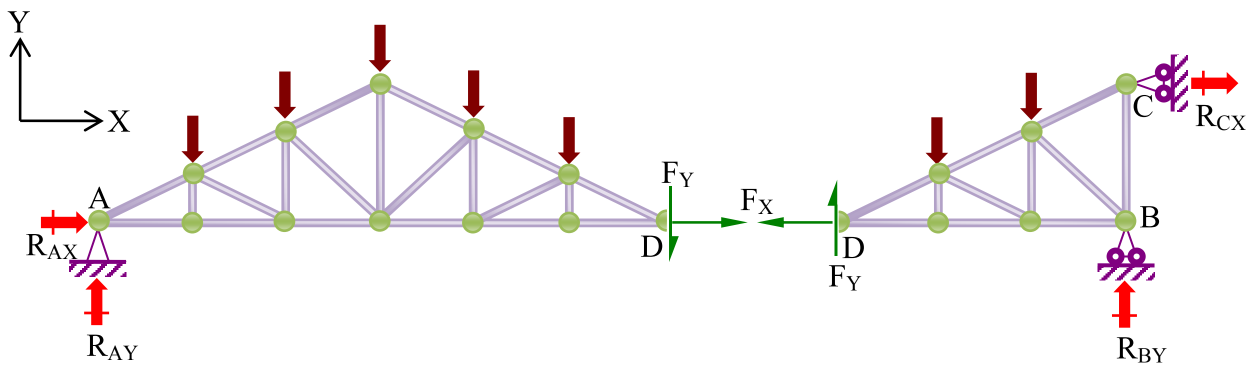


Figure 2.10: Schematic of two parts of the truss resulting from a cut at point D

2.4.4 Method of joints

For a truss consisting of r_a support reactions, n joints and m member, the total number of unknowns is $r_a + m$ and the total number of independent equilibrium equations that can be set up for all joints is $2n$ (two independent equilibrium equations can be set up for each joint). Since the degree of static indeterminacy $\text{DI} = (r_a + m) - 2n = 0$ for statically determinate trusses, the number of equilibrium equations at all joints is therefore sufficient for determining all unknown member forces and also support reactions.

The above idea constitutes a basis for the development of a well known technique, called a *method of joints*, for determining all member forces of statically determinate trusses. In this technique, each joint of the truss is first isolated from the structure and its corresponding free body diagram is then sketched (see Figure 2.11 for examples of FBDs for isolated joints).

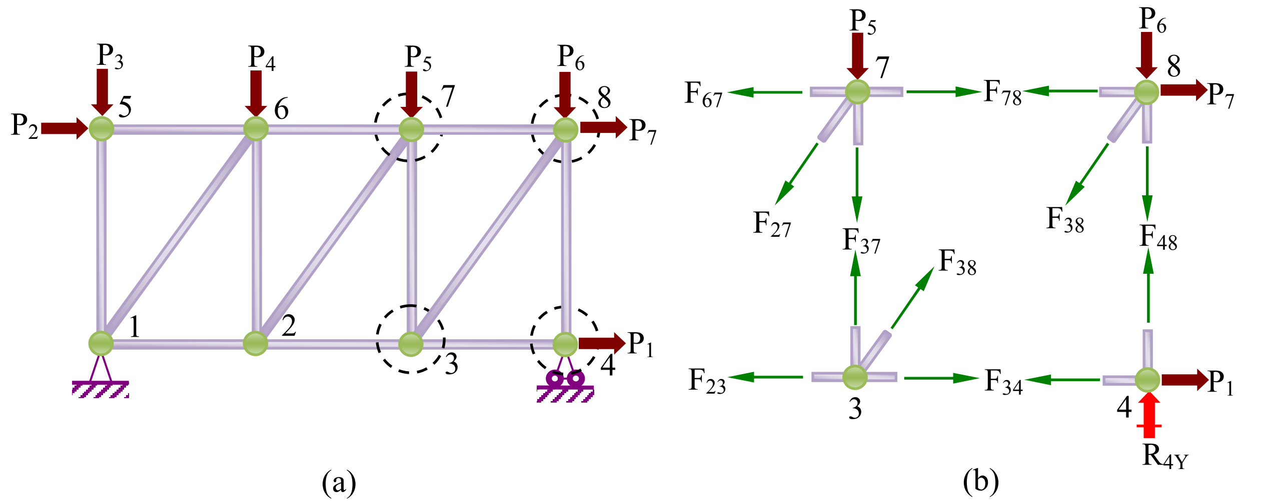


Figure 2.11: (a) Schematic of a 2D truss and (b) FBDs of joints 3, 4, 7 and 8

Since the internal force of any truss member consists of only the axial force, all forces acting to each joint (member forces and applied loads) constitute a system of concurrent forces. For two-dimensional trusses, two independent equilibrium equations can be set up for each joint, one corresponding to equilibrium of forces in the X-direction ($\Sigma F_X = 0$) and the other corresponding to equilibrium of forces in the Y-direction ($\Sigma F_Y = 0$). Once equilibrium equations of all joints are set up, such a system of linear equation can, in principle, be solved to obtain all member forces and support reactions.

To reduce computational effort especially when manual calculation is performed, we typically choose to avoid solving a large system of linear equations. Following guidelines can be useful when applied along with the method of joints.

- To prevent confusion and accidental errors, the member force is assumed a priori to be in tension in the sketch of joint FBD.
- Support reactions should be computed first by using a procedure stated in the section 2.4.3 in order to reduce the number of unknowns. However, this is not a must since a set of equations constructed at all joint is sufficient for determining both member forces and support reactions.
- Joint that consists of only two unknowns should be considered first since such unknowns can be solved by considering only equilibrium of that joint. Consider for example a truss shown in Figure 2.11. Once the support reactions $\{R_{1X}, R_{1Y}, R_{4Y}\}$ are computed, either joint 4 or joint 5 can be considered first since they contains only two unknowns $\{F_{15}, F_{56}\}$ and $\{F_{34}, F_{48}\}$, respectively. Let's say that we start with the joint 4. Once the member forces $\{F_{34}, F_{48}\}$ are obtained, joint 8 now becomes a good candidate for the next step since it contains only two unknowns $\{F_{38}, F_{78}\}$. This process can be repeated until there is no joint containing two unknowns. Determination of all support reactions before application of the method of joints, while is not necessary, increases the possibility to find joints that contain only two unknowns.
- If there exists a joint such that (1) it contains only two members, (2) the angle between those two members is not equal to 180 degrees, and (3) there is no applied load in both

directions, then the member forces of the two members vanish. This argument can readily be proved by considering equilibrium of that joint and the fact that a pair of nonparallel forces can be in equilibrium if and only if they vanish identically. For instance, a joint F in a truss shown in Figure 2.12 satisfies above conditions thus rendering the member forces F_{AF} and F_{FG} vanish (see also the FBD of the joint F in Figure 2.13 for clarity).

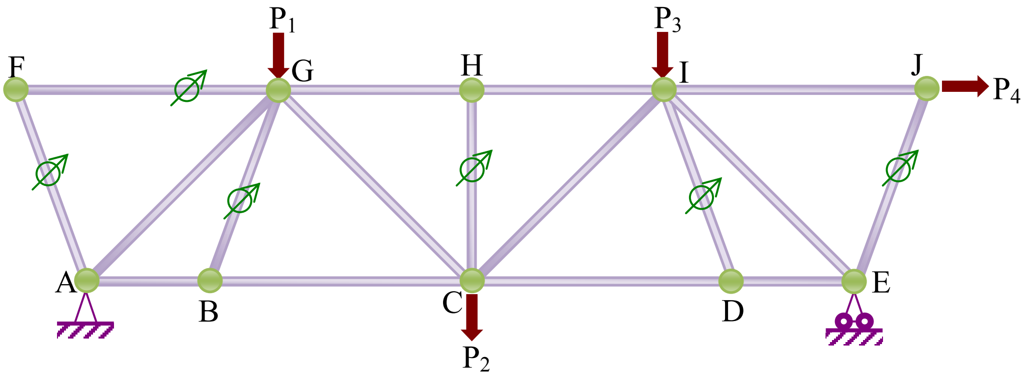


Figure 2.12: Schematic of truss containing members of zero member forces

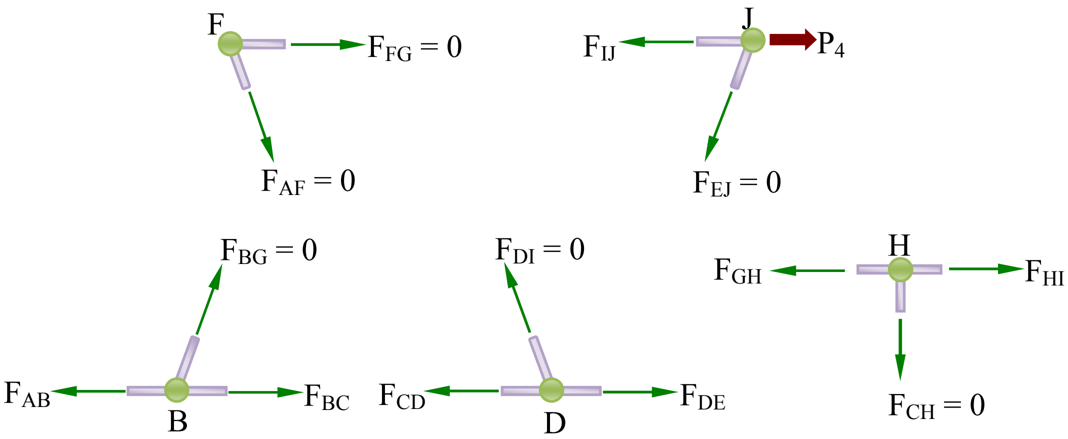


Figure 2.13: FBDs of joints B, D, F, H and J of truss shown in Figure 2.12

- If there exists a joint such that (1) it contains only two members, (2) the angle between those two members is not equal to 180 degrees, and (3) there is only one applied load or one component of the support reaction in the direction parallel to one member, then the member force of the other member vanishes. This argument can readily be proved by considering equilibrium of forces of that joint in the direction perpendicular to the applied load. For instance, a joint J in the truss shown in Figure 2.12 satisfies above conditions and this gives $F_{EJ} = 0$ (see also the FBD of the joint J in Figure 2.13 for clarity).
- If there exists a joint such that (1) it contains only three members, (2) two of the three members are parallel, and (3) there is no applied load at that joint, then the member force of the third member that are not parallel to the other two vanishes. Again, this argument can be verified by considering equilibrium of forces of that joint in the direction perpendicular to the two parallel members. For instance, joints B, D, and H in the truss shown in Figure 2.12 satisfies above conditions and this yields $F_{BG} = F_{DI} = F_{CH} = 0$ (see also the FBDs of the joint B, D, and H in Figure 2.13 for clarity).

- For a joint that all forces acting to that joint can be represented by a set of three forces, equilibrium of such joint implies that vectors of the three forces must form sides of a triangle. This feature allows magnitude of two of the three forces be obtained by *the law of sine* provided that magnitude of one force is known. From equilibrium of the joint, it implied in addition that the direction of each vector (as indicated by an arrow) must ensure a zero sum of the three vectors (i.e. the three arrows form a closed loop triangle). This latter property is sufficient for identifying direction of two forces provided that the direction of the other force is known.

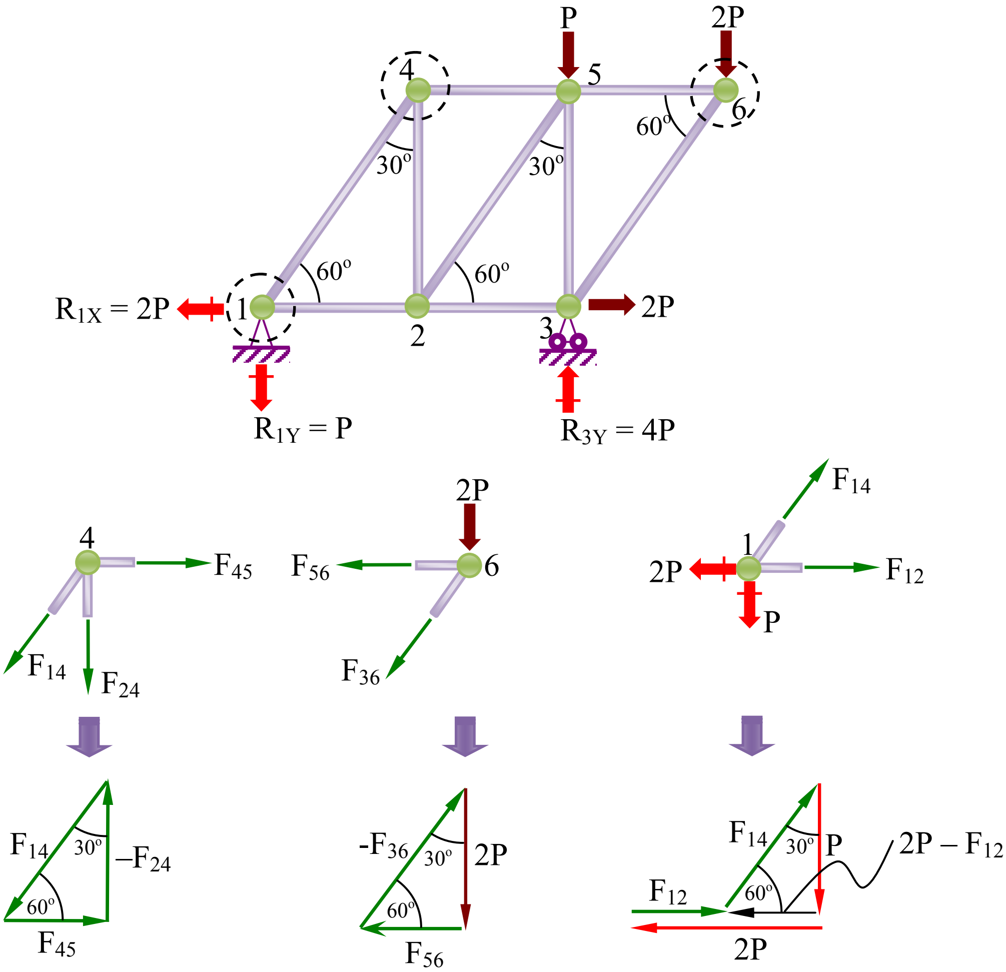


Figure 2.14: Schematic of a 2D truss and FBDs of joints 1, 4 and 6

For instance, consider a truss shown in Figure 2.14. A joint 6 contains only three forces (i.e. one applied load and two member forces) as shown in the FBD of this joint. From the law of sine, we obtain

$$\frac{-F_{36}}{\sin 90^o} = \frac{F_{56}}{\sin 30^o} = \frac{2P}{\sin 60^o}$$

From above equation, the two unknowns $\{F_{36}, F_{56}\}$ can readily be solved since the applied load is known. In addition, the members 36 and 56 must be in compression and tension, respectively, for the joint to be in equilibrium with the applied load 2P (see a triangle representing those three forces in Figure 2.14). Next, consider a joint 1 containing a pinned supports. While there are four forces acting at this joint (i.e. two reactions and two member forces), the support reaction 2P and the member force F_{12} are parallel and can be combined into one force as shown in Figure 2.15. Since reactions are known, the two unknown member forces $\{F_{12}, F_{14}\}$ can be obtained from the law of sine:

$$\frac{F_{14}}{\sin 90^o} = \frac{2P - F_{12}}{\sin 30^o} = \frac{P}{\sin 60^o}$$

From the diagram of forces, it is evident that both members 12 and 14 are in tension. Finally, let us consider a joint 4 that contains three member forces $\{F_{14}, F_{24}, F_{45}\}$ as shown by its FBD in Figure 2.14. Since the magnitude and direction of the member force F_{14} are already known from the joint 1, the members 24 and 45 must be in compression and in tension, respectively, and the magnitude of $\{F_{24}, F_{45}\}$ can be computed from the law of sine:

$$\frac{-F_{24}}{\sin 60^o} = \frac{F_{45}}{\sin 30^o} = \frac{F_{14}}{\sin 90^o}$$

Note that the negative sign appearing in $-F_{36}$, $-F_{12}$, and $-F_{24}$ is due to that the length of each side of the triangle must be non-negative and the member force is always positive in tension. Note also that the consideration of equilibrium of joints containing only three forces by representing them by sides of the triangle and then applying the law of sine provides an attractive alternative to that by solving the following two equilibrium equations of forces in X- and Y-directions, i.e. $\Sigma F_X = 0$ and $\Sigma F_Y = 0$.

- If all support reactions are determined before the method of joints is applied, there will be r_a equations exceeding the number of member forces. These equations can be used as a final check of support reactions and member forces already computed. Specially, if all support reactions and member forces are computed correctly, these r_a equations must be satisfied automatically; on the contrary, if some equilibrium equations are not satisfied, either support reactions or member forces are wrong.

As a final remark, the method of joints is a good candidate if all member forces are to be determined but, if only certain member forces are of interest, the method leads to a significant amount of effort associated with determination of other member forces. For instance, if only the member force F_{37} of the truss shown in Figure 2.11(a) is of interest, the joint 4 and the joint 8 must be considered first before the joint 3 can be treated. Another drawback of the method of joints becomes evident when the technique is applied to statically determinate trusses with all of their joints containing at least three members (e.g. trusses shown in Figure 2.15). For these particular structures, the method of joints leads to a large system of linear equations to be solved.

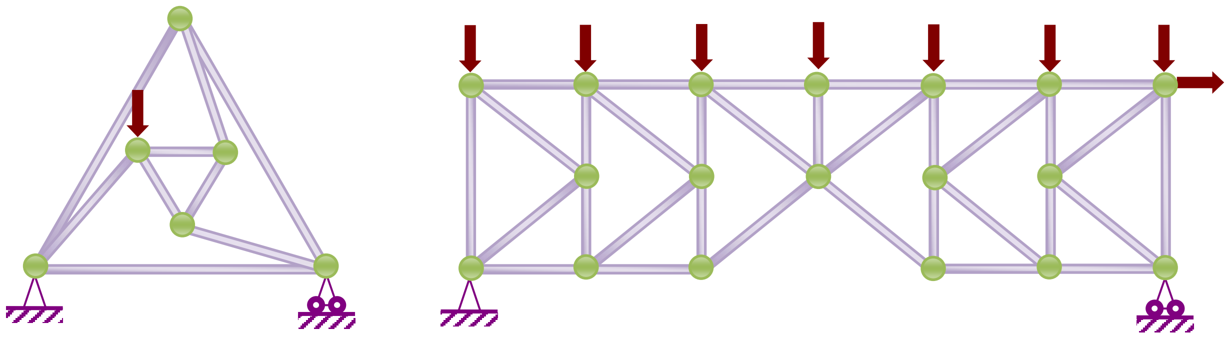
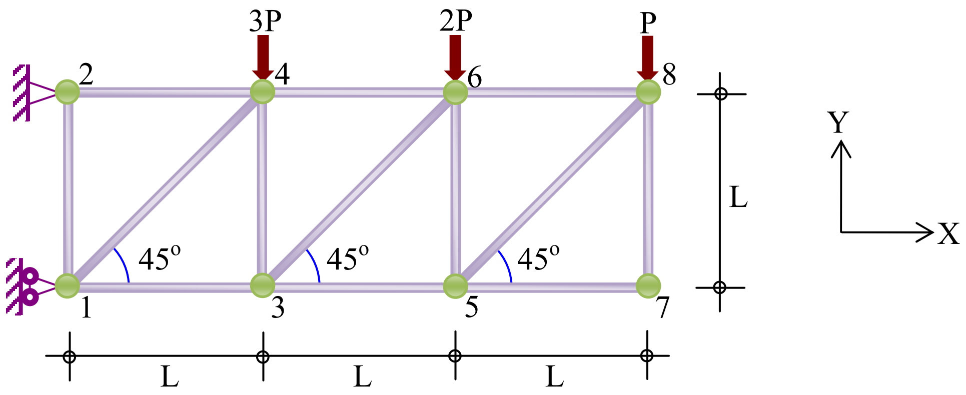
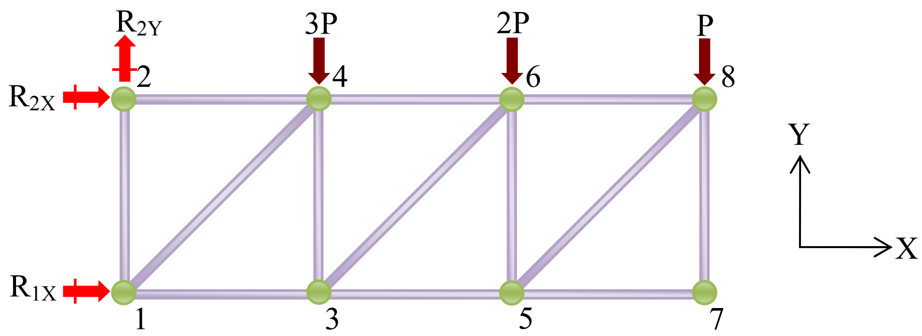


Figure 2.15: Example of determinate trusses with all joints containing at least three members

Example 2.4 Determine all support reactions and then compute all member forces for a truss shown below by the method of joints



Solution The structure given above is statically determinate (i.e. $r_a = 2 + 1 = 3$, $m = 13$, $n = 8$, $n_c = 0$, then $DI = 3 + 13(1) - 8(2) - 0 = 0$); thus all support reactions and member forces can be determined from static equilibrium. Since the number of support reactions is equal to 3, they can be computed by the consideration of equilibrium of the entire structure as shown below (the FBD of the entire structure is also shown below).



$[\Sigma F_Y = 0]$ $\uparrow +$: $R_{2Y} - 3P - 2P - P = 0$

$R_{2Y} = 6P$ Upward

$[\Sigma M_2 = 0]$ $\circlearrowleft +$: $(R_{1X})(L) - (3P)(L) - (2P)(2L) - (P)(3L) = 0$

$R_{1X} = 10P$ Rightward

$[\Sigma F_X = 0]$ $\rightarrow +$: $R_{1X} + R_{2X} = 0$

$R_{2X} = -10P$ Leftward

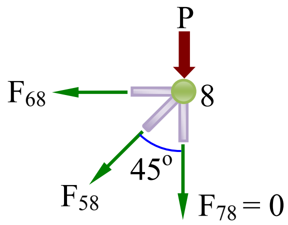
The member forces are then determined by the method of joints. Since the joint 7 contains only two non-parallel members and there is no applied load, it can be deduced that $F_{57} = F_{78} = 0$. Based on the known member forces $\{F_{57}, F_{78}\}$ and the known support reactions $\{R_{1X}, R_{2X}, R_{2Y}\}$, only joints 2 and 8 that contain only two unknowns. Let us start with the joint 8. The member forces $\{F_{58}, F_{68}\}$ can be obtained as follow:

$[\Sigma F_Y = 0]$ $\uparrow +$: $-F_{58}\cos 45^o - F_{78} - P = 0$

$F_{58} = -\sqrt{2}\,P$ (Compression)

$[\Sigma F_X = 0]$ $\rightarrow +$: $-F_{58}\sin 45^o - F_{68} = 0$

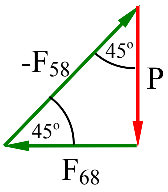
$F_{68} = P$ (Tension)



Since member force F_{58} and F_{68} are negative and positive, respectively, the members 58 and 68 are therefore in compression and in tension, respectively. Note that since the member force $F_{78} = 0$, the joint 8 contains only three non-zero forces and this allows the law of sine be alternatively used to determine the two unknown member forces $\{F_{58}, F_{68}\}$ as follow:

$$\frac{-F_{58}}{\sin 90^o} = \frac{F_{68}}{\sin 45^o} = \frac{P}{\sin 45^o} \Rightarrow F_{58} = -\sin 90^o \frac{P}{\sin 45^o} = -\sqrt{2}P \quad \text{(Compression)}$$

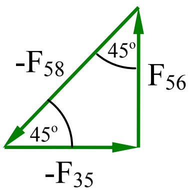
$$\Rightarrow F_{68} = \sin 45^o \frac{P}{\sin 45^o} = P \quad \text{(Tension)}$$



Once the member force F_{58} is determined, the joint 5 now contains only two unknown $\{F_{35}, F_{56}\}$. Since the member force $F_{57} = 0$, the joint 5 contains only three non-zero forces and they are shown in the diagram below. The unknowns $\{F_{35}, F_{56}\}$ are obtained as follow:

$$\frac{-F_{35}}{\sin 45^o} = \frac{F_{56}}{\sin 45^o} = \frac{-F_{58}}{\sin 90^o} \Rightarrow F_{35} = -\sin 45^o \frac{\sqrt{2}P}{\sin 90^o} = -P \quad \text{(Compression)}$$

$$\Rightarrow F_{56} = \sin 45^o \frac{\sqrt{2}P}{\sin 90^o} = P \quad \text{(Tension)}$$



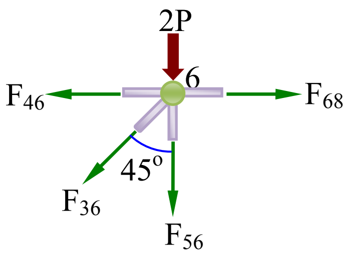
Once the member forces F_{56} and F_{68} are determined, the next joint to be considered is the joint 6 since it contains only two unknowns $\{F_{36}, F_{46}\}$. By considering a FBD of the joint 6 and then setting two equilibrium equations, we obtain

$[\Sigma F_Y = 0]$ $\uparrow +$: $-F_{36}\cos 45^o - F_{56} - 2P = 0$

$F_{36} = -3\sqrt{2}\,P$ (Compression)

$[\Sigma F_X = 0]$ $\rightarrow +$: $-F_{36}\sin 45^o - F_{46} + F_{68} = 0$

$F_{46} = 4P$ (Tension)



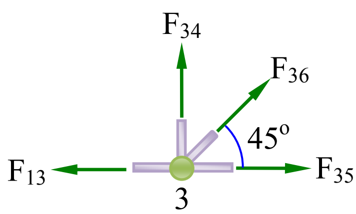
Once the member forces F_{35} and F_{36} are determined, the next joint to be considered is the joint 3 since it contains only two unknowns $\{F_{36}, F_{46}\}$. By considering a FBD of the joint 3 and then setting two equilibrium equations, we obtain

$[\Sigma F_X = 0]$ $\rightarrow +$: $F_{36}\cos 45^o + F_{35} - F_{13} = 0$

$F_{13} = -4P$ (Compression)

$[\Sigma F_Y = 0]$ $\uparrow +$: $F_{36}\sin 45^o + F_{34} = 0$

$F_{34} = 3P$ (Tension)



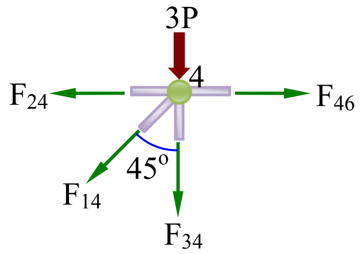
Once the member forces F_{34} and F_{46} are determined, the next joint to be considered is the joint 4 since it contains only two unknowns $\{F_{14}, F_{24}\}$. By considering a FBD of the joint 4 and then setting two equilibrium equations, we obtain

$[\Sigma F_Y = 0]$ $\uparrow +$: $-F_{14}\cos 45^o - F_{34} - 3P = 0$

$F_{14} = -6\sqrt{2}\,P$ (Compression)

$[\Sigma F_X = 0]$ $\rightarrow +$: $-F_{14}\sin 45^o - F_{24} + F_{46} = 0$

$F_{24} = 10P$ (Tension)



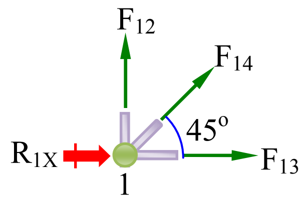
Once the member forces F_{13} and F_{14} are determined, the next joint to be considered is the joint 1 since it contains only one unknown $\{F_{12}\}$. By considering a FBD of the joint 1 and then setting two equilibrium equations, we obtain

$[\Sigma F_Y = 0]$ $\uparrow +$: $F_{14}\sin 45^o + F_{12} = 0$

$F_{12} = 6P$ (T)

$[\Sigma F_X = 0]$ $\rightarrow +$: $F_{14}\cos 45^o + F_{13} + R_{1X} = 0$?

$(-6\sqrt{2}P)(1/\sqrt{2}) - 4P + 10P = 0$ OK



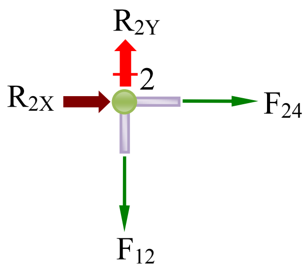
It is evident that the second equation is not needed in the calculation of member forces and, in addition, there are still two equations left at joint 2. This is not a surprise since three equilibrium equations associated with the entire structure were already employed in the calculation of support reactions. As a result, the three equilibrium equations (one at joint 1 and the other two at joint 2) constitute no new independent equation but, in fact, they are linear combinations of other equilibrium equations established above. Although they are not required in the calculation of member forces, such extra equations are still useful as a part of verification of calculated results; if all member forces are computed accurately, they must automatically satisfy those equations. For instance, two equilibrium equations at joint 2 must also be satisfied as follow:

$[\Sigma F_Y = 0]$ $\uparrow +$: $R_{2y} - F_{12} = 0$?

: $6P - 6P = 0$ OK

$[\Sigma F_X = 0]$ $\rightarrow +$: $R_{2x} + F_{24} = 0$?

: $-10P - 10P = 0$ OK



2.4.5 Method of sections

As already pointed out in the previous section, the method of joints becomes inefficient if the internal forces of certain members are of interest. To further enhance the efficiency of computation of member forces of statically determinate trusses, another technique called a *method of sections* is introduced. This method simply employs static equilibrium conditions along with the method of structure partitioning.

To outline and clearly demonstrate the method of sections, let us consider a truss shown in Figure 2.16(a). Assume that the member forces $\{F_{23}, F_{27}, F_{67}\}$ are of interest. As a first step, all support reactions $\{R_{1X}, R_{1Y}, R_{4Y}\}$ are determined via the consideration of equilibrium of the entire truss. To access and see the member forces $\{F_{23}, F_{27}, F_{67}\}$, we need to introduce a fictitious cut or a section passing through the members 23, 27 and 67 as shown by a dash line in Figure 2.16(a). This cut not only exposes the member forces $\{F_{23}, F_{27}, F_{67}\}$ but also divides the structure into two parts with the FBD of each part shown in Figure 2.16(b) and 2.16(c). As is evident, either the FBD of the right part or the FBD of the left part contains exactly three unknowns $\{F_{23}, F_{27}, F_{67}\}$. Thus, the consideration of equilibrium of either one of those two parts yields a sufficient number of equations for solving the three unknown member forces. For instance, enforcing equilibrium of moments about joint 7 of the right part yields the member force F_{23}; enforcing equilibrium of moments about joint 2 of the right part yields the member force F_{67}; and enforcing equilibrium of forces in the Y-direction of the right part yields the member force F_{27}. It is worth noting that other equivalent sets of three equilibrium equations can also be used and that the left part of the truss can also be employed to determine $\{F_{23}, F_{27}, F_{67}\}$. Note, however, that the consideration of equilibrium of those two parts, while given totally six equations, still yields only three independent equations since the other three equations are equivalent to those employed for computing the support reactions $\{R_{1X}, R_{1Y}, R_{4Y}\}$.

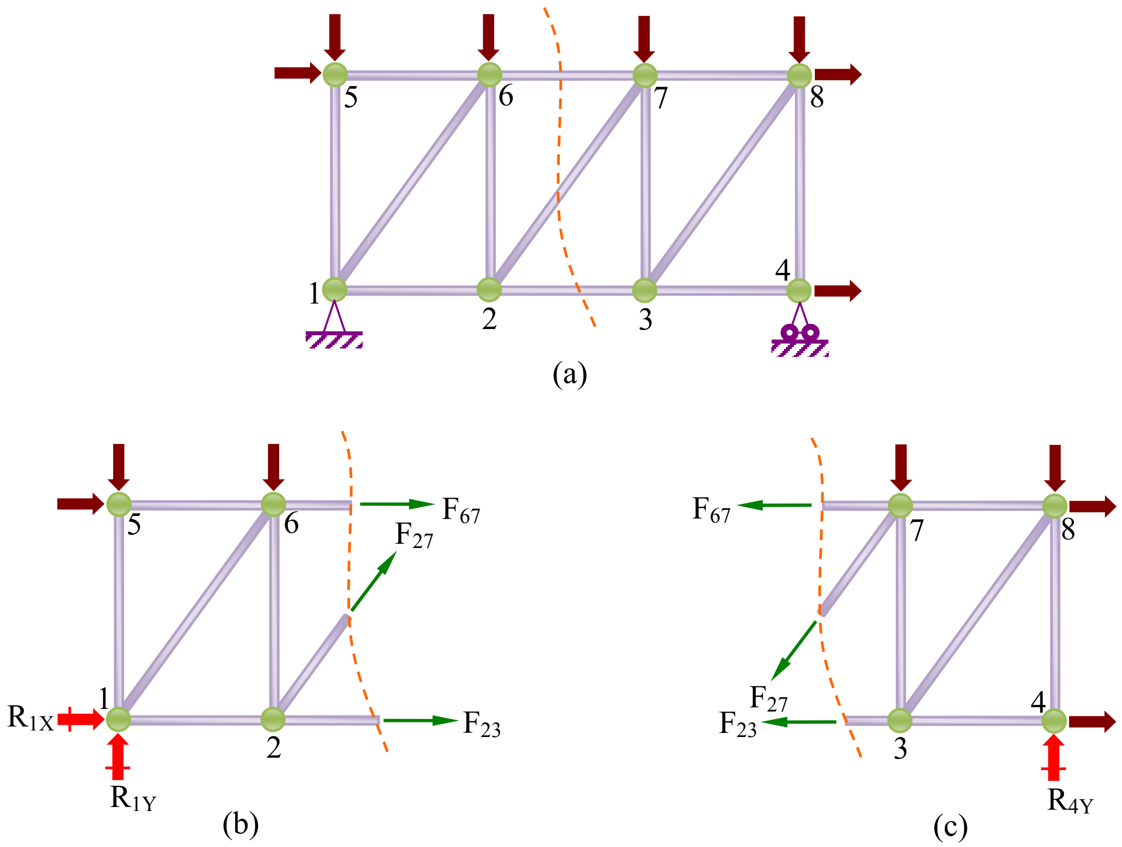


Figure 2.16: (a) Schematic of a truss and location of fictitious cut (b) & (c) FBDs of parts resulting from partitioning

If the member force F_{37} is of interest, we may introduce a fictitious cut as shown in Figure 2.17(a). This cut passes through the member 37 and divides the structure into two parts with the corresponding FBDs shown in Figure 2.17(b) and (c). It is obvious that both FBDs contain exactly three unknown member forces $\{F_{23}, F_{37}, F_{78}\}$ and they can in principle be determined from equilibrium equations set up for one of these two parts. To determine the member force F_{37}, we simply enforce equilibrium of forces in Y-direction of the left part. If the member forces $\{F_{23}, F_{78}\}$

are also of interest, they can be obtained by enforcing equilibrium of moments about joint 7 and joint 3, respectively.

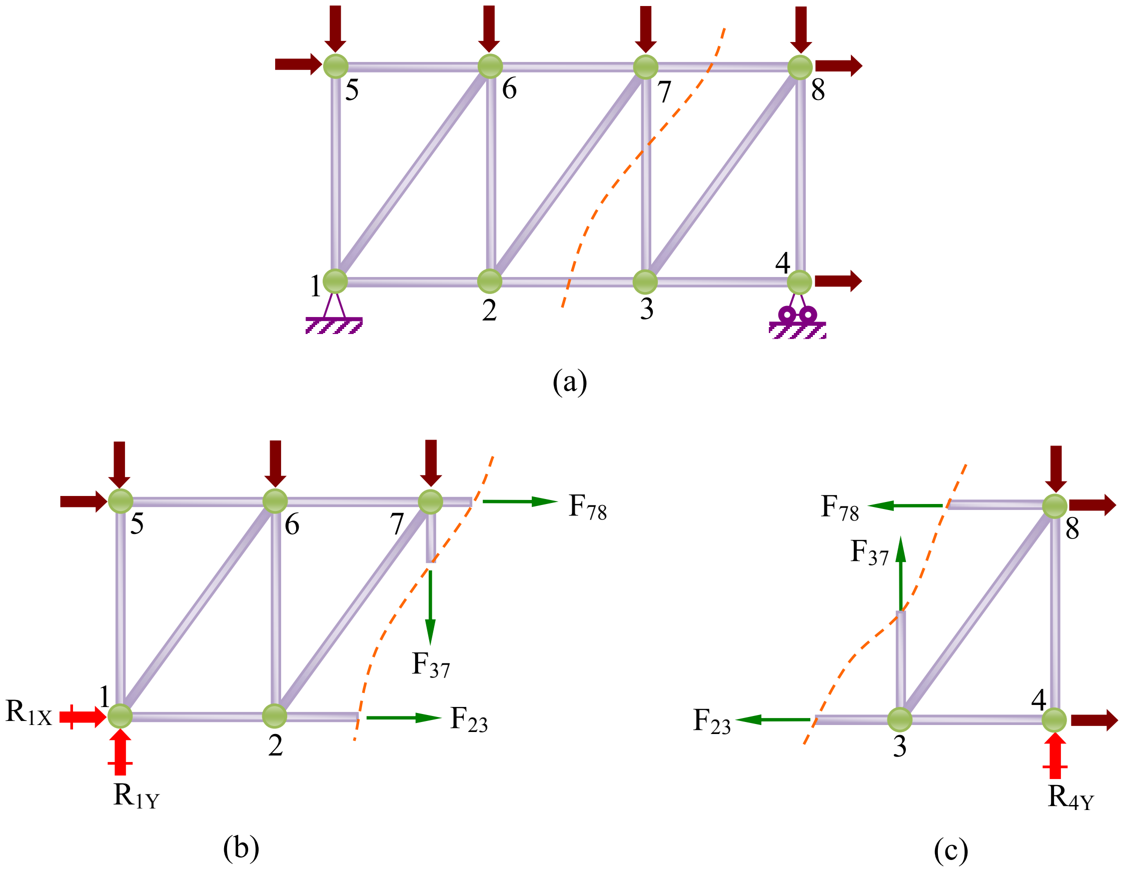


Figure 2.17: (a) Schematic of a truss and location of fictitious cut and (b) & (c) FBDs of parts resulting from partitioning

To apply the method of sections in an efficient manner, following remarks may be taken into account to reduce computational effort:

- Since the internal force is constant throughout the member, the location through which the section passes does not affect the results. This therefore provides flexibility for choosing a path for sectioning.
- To prevent confusion and errors, it is generally assumed a positive sign convention for all unknown member forces (i.e. members are assumed in tension). If the negative member force is obtained, the member is therefore in compression.
- All support reactions should be determined before the method of sections is applied in order to reduce the number of unknowns appearing in any parts resulting from the sectioning.
- If a section introduces only three unknown member forces along the cut, such unknown forces can be determined from equilibrium of either one of the two parts provided that all forces are not concurrent forces, e.g. sections shown in Figure 2.16 and Figure 2.17.
- If a section and a reference point used for taking the moment are chosen appropriately, the member force can readily be obtained from equilibrium of moments. For instance, if the member force F_{12} of trusses shown in Figure 2.18 is to be computed, the section and reference point A may be chosen as shown below.

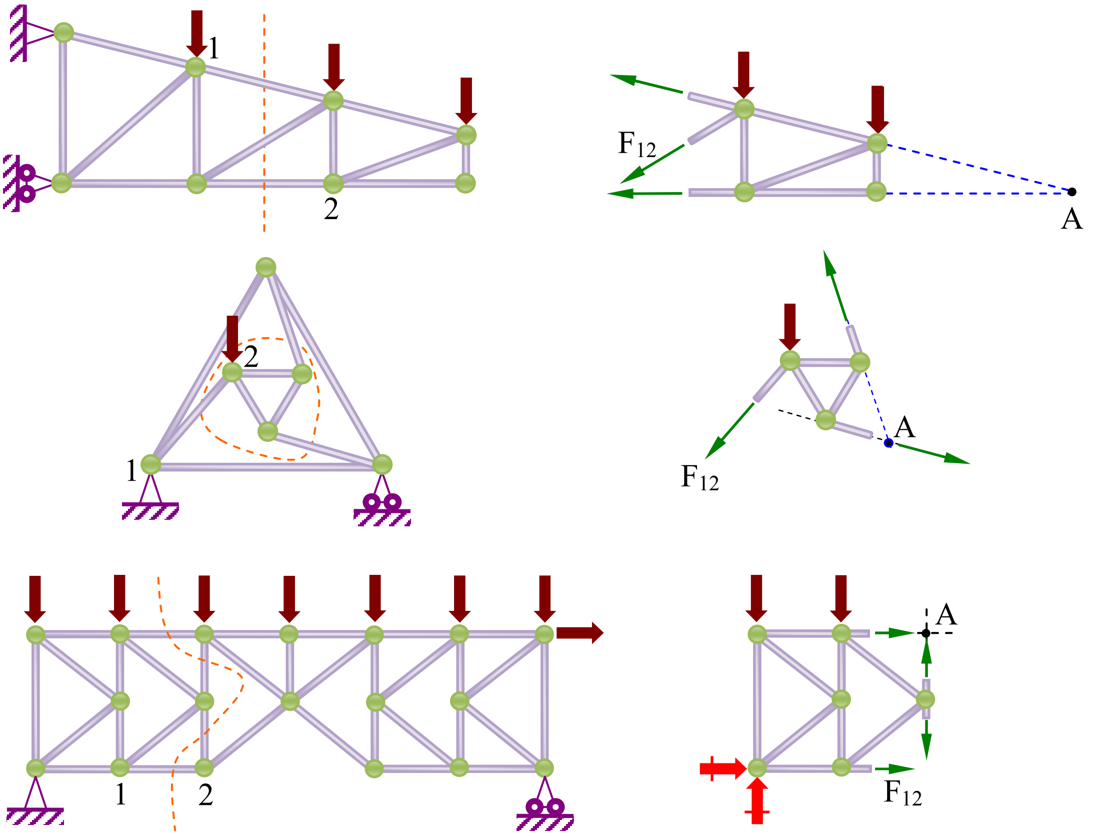
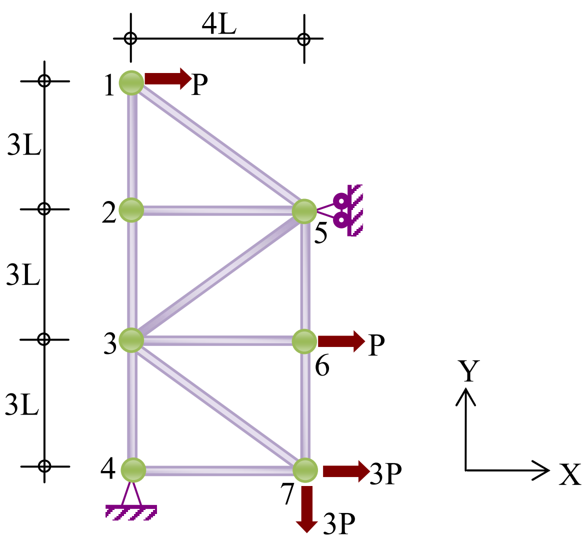


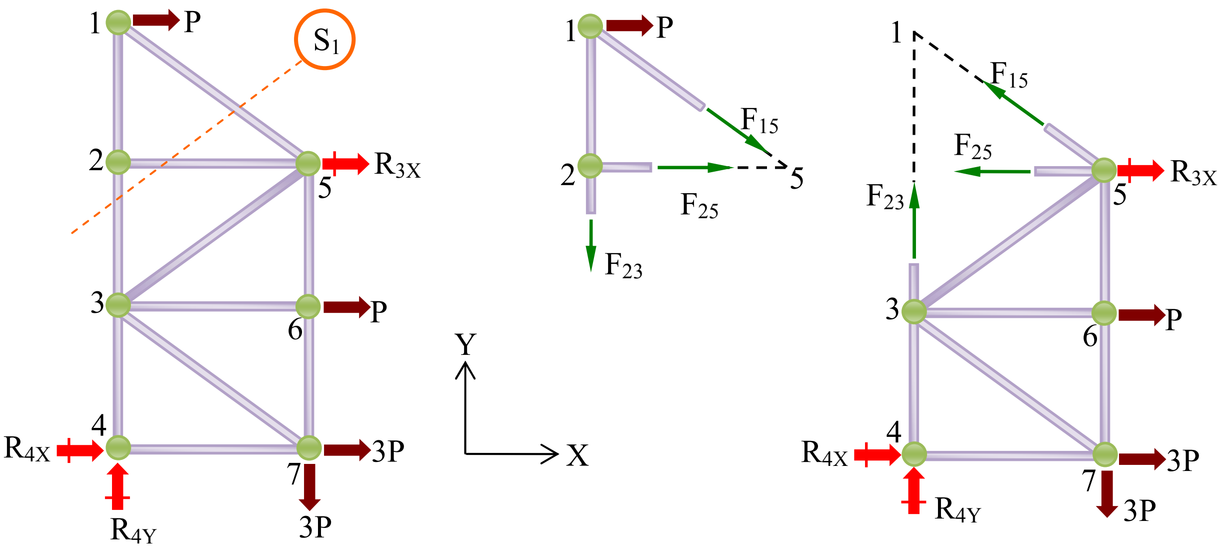
Figure 2.18: Schematic of trusses, sections, FBD of parts of truss, and reference point A

The method of sections has been found more efficient than the method of joints when the internal forces are to be determined for certain members. However, for certain trusses, both methods may be used together to increase the efficiency. For instance, in the analysis of the last two trusses shown in Figure 2.18 (every joint of these truss contains at least three members), the method of sections is applied first to determine some member forces. The method of joints can subsequently be used to compute all remaining member forces in a simple fashion since it is always possible to find joints containing only two unknowns.

Example 2.5 Determine all support reactions and then use the method of sections to compute the member forces F_{23}, F_{25}, F_{35}, and F_{56} of a truss shown below



<u>**Solution**</u> The structure given above is statically determinate (i.e. $r_a = 2 + 1 = 3$, $m = 11$, $n = 7$, $n_c = 0$, then $DI = 3 + 11(1) - 7(2) - 0 = 0$); thus all support reactions and member forces can be determined from static equilibrium. Since the number of support reactions is equal to 3, they can be computed by considering equilibrium of the entire structure as shown below.



$[\Sigma F_Y = 0]$ $\uparrow +$: $R_{4Y} - P = 0$

$R_{4Y} = 3P$ Upward

$[\Sigma M_4 = 0]$ $\circlearrowleft +$: $-(R_{3X})(6L) - (P)(9L) - (P)(3L) - (3P)(4L) = 0$

$R_{3X} = -4P$ Leftward

$[\Sigma F_X = 0]$ $\rightarrow +$: $R_{4X} + R_{3X} + P + P + 3P = 0$

$R_{4X} = -P$ Leftward

Next, by introducing a fictitious cut S_1 and then considering equilibrium of the top part of the truss, the member forces F_{23} and F_{25} can be obtained as follow:

$[\Sigma M_5 = 0]$ $\circlearrowleft +$: $(F_{23})(4L) - (P)(3L) = 0$

$F_{23} = 3P/4$ (Tension)

$[\Sigma M_1 = 0]$ $\circlearrowleft +$: $(F_{25})(3L) = 0$

$F_{25} = 0$

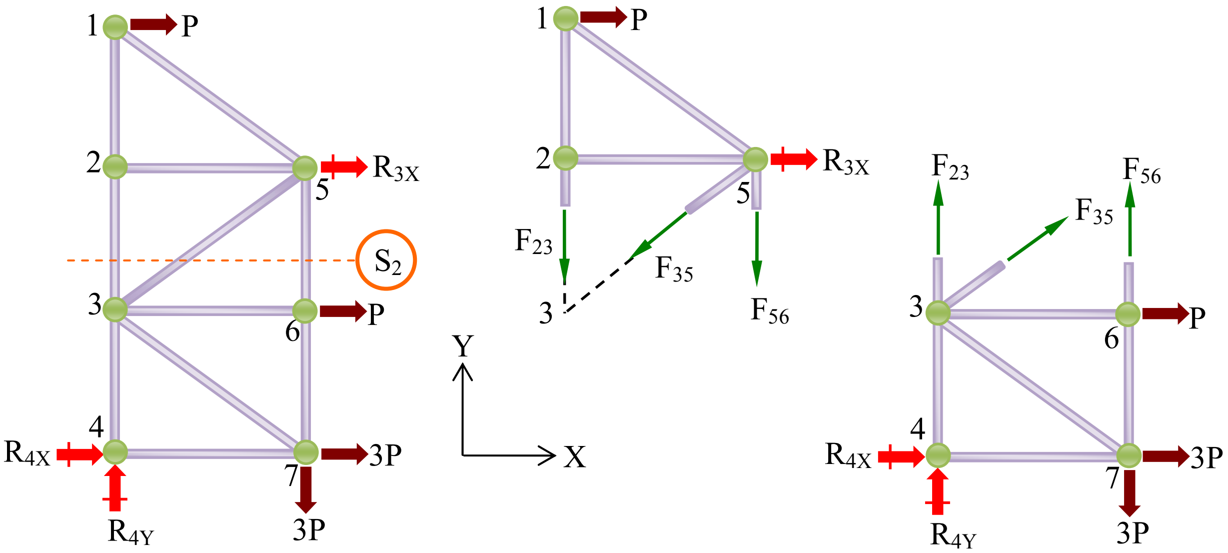
Alternatively, the member forces F_{23} and F_{25} can also be obtained by considering equilibrium of the bottom part of the truss as shown below:

$[\Sigma M_5 = 0]$ $\circlearrowleft +$: $P(3L) + (3P + R_{4X})(6L) - (F_{23} + R_{4Y})(4L) = 0$

$F_{23} = 3P/4$ (Tension)

$[\Sigma M_1 = 0]$ $\circlearrowleft +$: $(R_{3X} - F_{25})(3L) + (3P + R_{4X})(9L) + P(6L) - (3P)(4L) = 0$

$F_{25} = 0$



Next, by introducing another fictitious cut S_2 and again considering equilibrium of the top part of the truss, the member forces F_{35} and F_{56} can then be obtained as follow:

$[\Sigma M_3 = 0]$ ↺+ : $-(F_{56})(4L) - (R_{3X})(3L) - (P)(6L) = 0$

$F_{56} = 3P/2$ (Tension)

$[\Sigma F_Y = 0]$ ↑+ : $-(F_{35})(4/5) + P + R_{3X} = 0$

$F_{35} = -15P/4$ (Compression)

Similar to the previous case, the member forces F_{35} and F_{56} can equivalently be obtained by considering equilibrium of the bottom part of the truss as shown below:

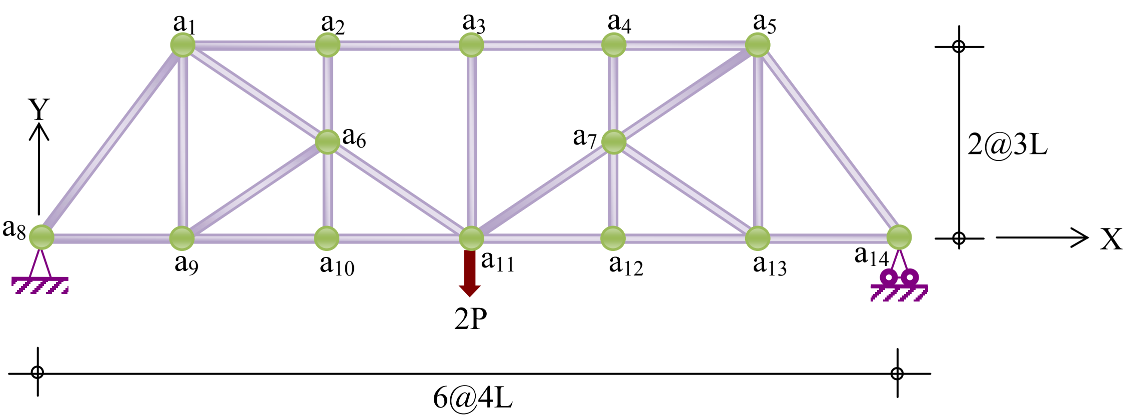
$[\Sigma M_3 = 0]$ ↺+ : $(F_{56} - 3P)(4L) + (3P + R_{4X})(3L) = 0$

$F_{56} = 3P/2$ (Tension)

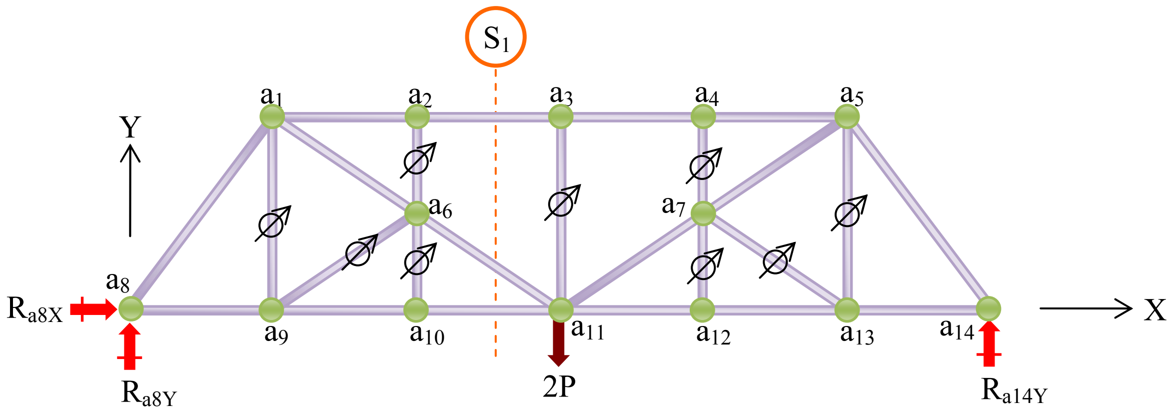
$[\Sigma F_X = 0]$ →+ : $(F_{35})(4/5) + P + 3P + R_{4X} = 0$

$F_{35} = -15P/4$ (Compression)

Example 2.6 Determine all support reactions and all member forces for a truss shown below



Solution The structure given above is statically determinate (i.e. $r_a = 2 + 1 = 3$, $m = 25$, $n = 14$, $n_c = 0$, then $DI = 3 + 25(1) - 14(2) - 0 = 0$); thus, all support reactions and member forces can be determined from static equilibrium. Since the number of support reactions is equal to 3, they can be computed by considering equilibrium of the entire structure as shown below.



$[\Sigma F_X = 0]$ $\rightarrow +$: $R_{a8X} = 0$

$[\Sigma M_{a8} = 0]$ $\circlearrowleft +$: $(R_{a14Y})(24L) - (2P)(12L) = 0$

$R_{a14Y} = P$

$[\Sigma F_Y = 0]$ $\uparrow +$ $R_{a14Y} + R_{a8Y} - 2P = 0$

$R_{a8Y} = P$

From geometry and loading of a given truss, it can readily be verified that the following member forces vanish:

$F_{a2a6} = F_{a3a11} = F_{a4a7} = F_{a6a10} = F_{a6a9} = F_{a1a9} = F_{a7a12} = F_{a7a13} = F_{a5a13} = 0$

From symmetry of geometry and loading, it can be deduced that

$F_{a1a8} = F_{a5a14}$; $F_{a8a9} = F_{a13a14}$; $F_{a1a2} = F_{a4a5}$; $F_{a1a6} = F_{a5a7}$

$F_{a9a10} = F_{a12a13}$; $F_{a2a3} = F_{a3a4}$; $F_{a6a11} = F_{a7a11}$; $F_{a10a11} = F_{a11a12}$

Next, by applying the method of joints to joints a_2, a_9, a_6 and a_{10}, it leads to

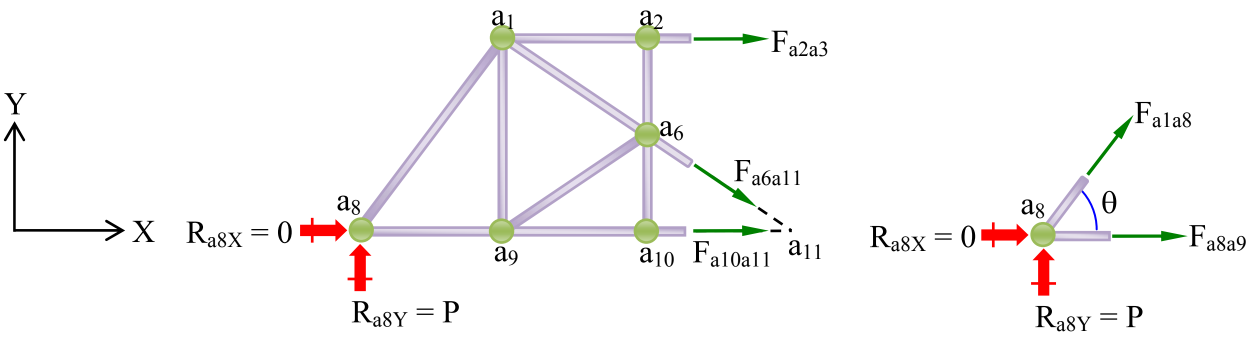
Joint a_2: $[\Sigma F_X = 0]$ $\Rightarrow$ $F_{a1a2} = F_{a2a3}$

Joint a_9: $[\Sigma F_X = 0]$ $\Rightarrow$ $F_{a8a9} = F_{a9a10}$

Joint a_6: $[\Sigma F_X = 0]$ $\Rightarrow$ $F_{a1a6} = F_{a6a11}$

Joint a_{10}: $[\Sigma F_X = 0]$ $\Rightarrow$ $F_{a9a10} = F_{a10a11}$

Now, it still remains to determine the member forces F_{a2a3}, F_{a6a11}, F_{a10a11} and F_{a1a8}. To compute the member forces F_{a2a3}, F_{a6a11} and F_{a10a11}, we apply the method of sections along with introducing a fictitious cut S_1 that passes through the members a_2a_3, a_6a_{11} and $a_{10}a_{11}$ as indicated in the above figure and detail calculations are given below.



$[\Sigma M_{a1} = 0]$ ↺+ : $(F_{a10a11})(6L) - (P)(4L) = 0$

$F_{a10a11} = 2P/3$ (Tension)

$[\Sigma M_{a11} = 0]$ ↺+ : $-(F_{a2a3})(6L) - (P)(12L) = 0$

$F_{a2a3} = -2P$ (Compression)

$[\Sigma F_Y = 0]$ ↑+ : $-(F_{a6a11})(3/5) + P = 0$

$F_{a6a11} = 5P/3$ (Tension)

Finally, the member force F_{a1a8} is obtained by applying the method of joints to joint a_8 as shown below ($\sin\theta = 3/\sqrt{13}$ and $\cos\theta = 2/\sqrt{13}$).

$[\Sigma F_Y = 0]$ ↑+ : $F_{a1a8} \sin\theta + P = 0$

$F_{a1a8} = -\sqrt{13}P/3$ (Compression)

$[\Sigma F_X = 0]$ →+ : $F_{a1a8} \cos\theta + F_{a8a9} = 0$?

$(-\sqrt{13}P/3)(2/\sqrt{13}) + 2P/3 = 0$ <u>OK</u>

Note that the second equation is just an extra equation that can be used to partially verify computed results. All member forces are summarized below:

$F_{a1a8} = F_{a5a14} = -\sqrt{13}P/3$ (Compression)

$F_{a8a9} = F_{a13a14} = F_{a9a10} = F_{a10a11} = F_{a11a12} = F_{a12a13} = 2P/3$ (Tension)

$F_{a1a2} = F_{a2a3} = F_{a3a4} = F_{a4a5} = -2P$ (Compression)

$F_{a1a6} = F_{a6a11} = F_{a5a7} = F_{a7a11} = 5P/3$ (Tension)

$F_{a1a6} = F_{a6a11} = F_{a5a7} = F_{a7a11} = 5P/3$ (Tension)

$F_{a1a9} = F_{a2a6} = F_{a6a9} = F_{a6a10} = F_{a3a11} = 0$

$F_{a4a7} = F_{a7a12} = F_{a5a13} = F_{a7a13} = 0$

2.5 Static Analysis of Beams

This section devotes to the static analysis of another type of structures termed *beams*. The primary objective is to present basic techniques commonly used in the determination of support reactions and the internal forces at any location, i.e. the shear force diagram (SFD) and the bending moment diagram (BMD). In addition, sketch of a qualitative elastic or deformed shape of beam under external applied loads is also discussed. The section starts with a brief introduction on characteristics of beams and standard notations and sign convention commonly used for beam. Next, we present a basic technique based upon the method of sections to determine the internal forces at a particular location of interest. A more general technique based on the differential and integral formula is further introduced to construct the SFD and BMD. Finally, guidelines useful for sketching the elastic curve are summarized. Various examples are also presented to demonstrate the principle and details of each technique.

2.5.1 Characteristics of beams

An idealized structure is called a *beam* if and only if (i) all members are straight and form a straight-line-configuration structure, (ii) all members are generally connected by beam joints (full moment release is allowed for certain joints), and (iii) all applied loads must form a system of transverse loads. Examples of beams are shown schematically in Figure 2.19. Note that there is no restriction on the type of supports present in beams, i.e. roller supports, pinned supports, guide supports and fixed supports are allowed. Note, however, that the component of support reactions in the direction of the beam axis, if exists, can be ignored since all applied loads are transverse loads.

From above definition, it can readily be verified that the internal forces at any location of the beam can be represented by two components called the *shear force* and the *bending moment*. The shear force is the resultant force in the direction perpendicular to the axis of the beam and the bending moment if the resultant moment in the direction perpendicular to the plane of transverse loads, see Figure 2.20. Unlike the truss member, the shear force and bending moment are in general not constant throughout the member. Another different feature is that the angle between the two members connecting to any beam joint is preserved before and after undergoing deformation provided that such a joint contains no moment release.

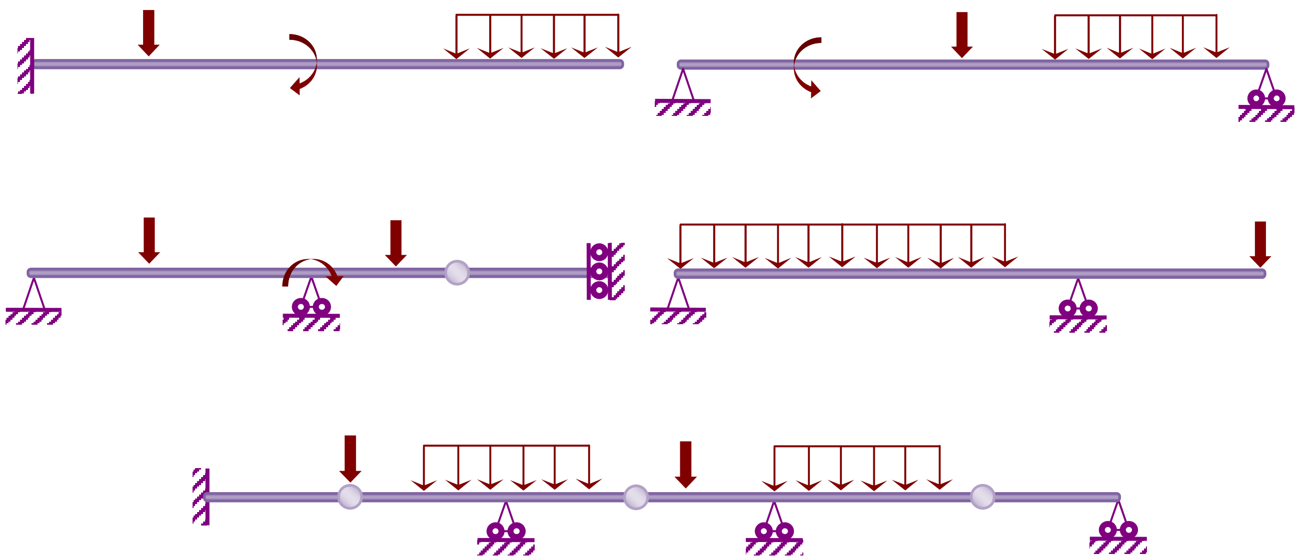


Figure 2.19: Schematic of some statically determinate beams

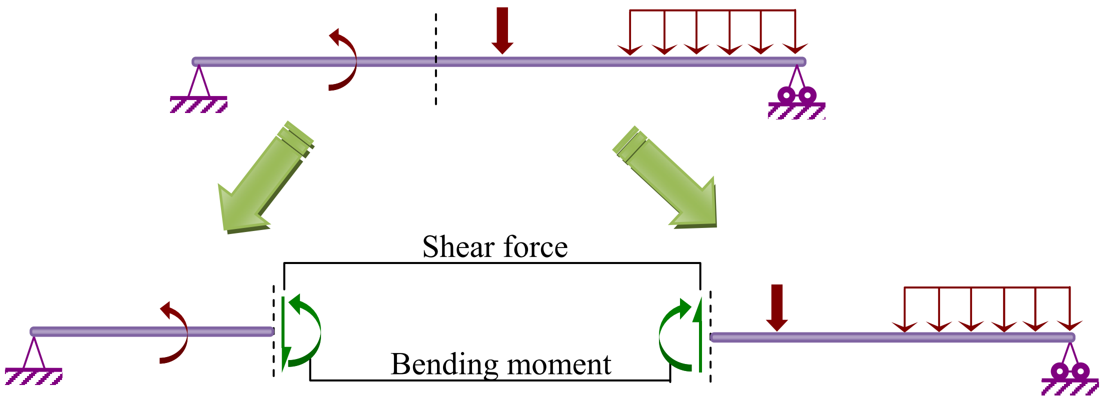


Figure 2.20: Schematic indicating two components of the internal force in beam

2.5.2 Sign and convention

A reference Cartesian coordinate system commonly used for beam, denoted by {O; X, Y, Z} where O is the origin and {X, Y, Z} represents three mutually perpendicular axes following a standard right handed rule, is defined such that the X-axis directs along the axis of the beam and the X-Y plane is a plane of transverse loads. Note that there is no restriction on the location of the origin O. An example of the reference coordinate system for a beam is shown in Figure 2.21.

The sign convention and notations for support reactions of beams can be defined in a similar fashion to those for support reactions of trusses. For instance, reactions at the fixed support of the beam shown in Figure 2.21 are denoted by R_{AY} and R_{AM} where the former stands for a force reaction in the Y-direction and the latter stands for a moment reaction in the Z-direction, and a force reaction in the Y-direction of the roller support located at a point B is denoted by R_{BY}. Since all support reactions are unknown a priori, it is common in the analysis to assume that they possess a positive direction, i.e. they direct along the positive coordinate axes (see examples of positive sign convention for reactions in Figure 2.21). Once results are obtained, the actual direction of each reaction can be decided from their sign; specifically, if the computed reaction is positive, the assumed direction is correct but, if the computed reaction is negative, the actual direction is opposite to the assumed direction.

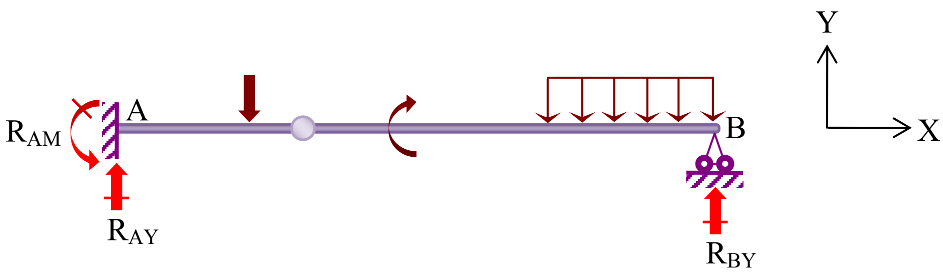


Figure 2.21: Schematic showing a reference coordinate system and sign convention and notations of support reactions of a beam

For the shear force and bending moment, it is standard to follow notations and sign convention given below.

- The shear force at a particular point A is denoted by a symbol V_A and the shear force as a function of position x along the beam is denoted by V(x). The shear force at any point is considered to be positive if and only if it tends to produce a clockwise rotation of the infinitesimal beam element in the neighborhood of that point; otherwise it is negative. The positive and negative shear forces are shown in Figure 2.22. A traditional strategy commonly used to memorize the sign convention for the shear force is that "*the shear force at any point is considered positive if and only if, when a cut is made at that point, the shear force directs upward in the FBD of the right part of a beam and directs downward in the FBD of the left part of a beam*".

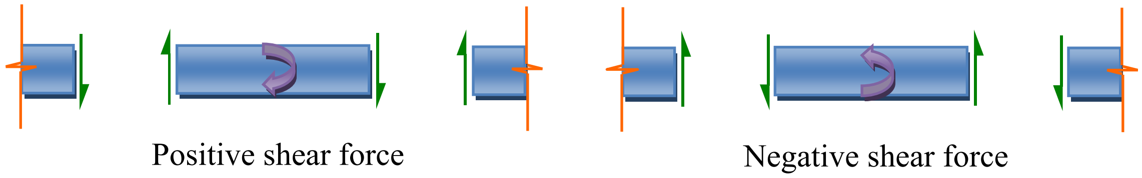


Figure 2.22: Schematic indicating positive and negative sign convention for shear force

- The bending moment at a particular point A is denoted by a symbol M_A and the bending moment as a function of position x along the beam is denoted by M(x). The bending moment at any point is considered to be positive if and only if it produces a compressive stress at the top and produces a tensile stress at the bottom; otherwise it is negative. The positive and negative bending moments are shown in Figure 2.23. A traditional strategy commonly used to memorize the sign convention for the bending moment is that "*the positive bending moment produces a concave upward curve or a smile shape while the negative moment produces a concave downward curve or a sad shape*".

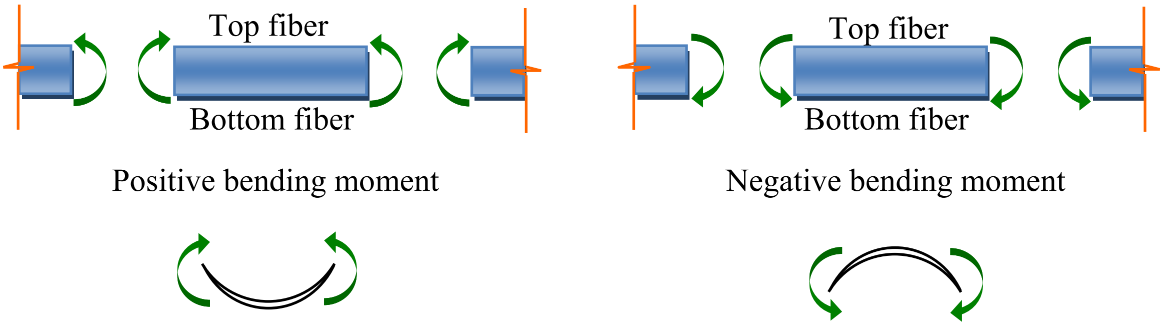


Figure 2.23: Schematic indicating positive and negative sign convention for bending moment

2.5.3 Determination of support reactions

A standard procedure for determining support reactions of statically determinate beams follows exactly that given in the section 2.3. For a beam containing only two components of support reactions, consideration of equilibrium of the entire beam is sufficient for solving all unknown reactions. For instance, support reactions {R_{AY}, R_{BY}} of a simply-supported beam shown in Figure 2.24 can be obtained by solving two equilibrium equations set up on the entire beam as follows:

- the reaction R_{AY} is obtained from equilibrium of moments of the entire structure about a point A, and
- the reaction R_{BY} is obtained either from equilibrium of forces in Y-direction of the entire structure or from equilibrium of moments of the entire structure about a point B.

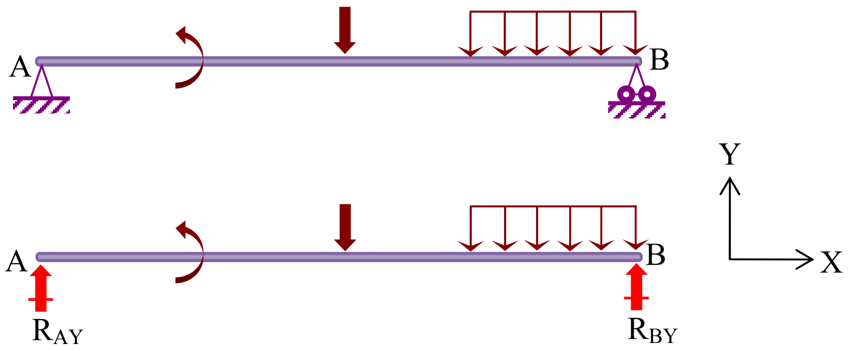


Figure 2.24: Schematic of a simply-supported beam and its FBD

For various beams, they may contain more than two components of support reactions while still statically determinate, for instance, the third beam shown in Figure 2.19, the beam shown in Figure 2.21 and the beam shown in Figure 2.25. For this particular case, only the consideration of equilibrium of the entire beam is insufficient to determine all support reactions. Additional conditions related to the presence of internal releases within the beam must be employed to supply adequate number of equations. These extra equations can equivalently be viewed as equilibrium equations written for some parts of the beam that resulting from suitable cuts (e.g. cuts passing through the location of the internal releases).

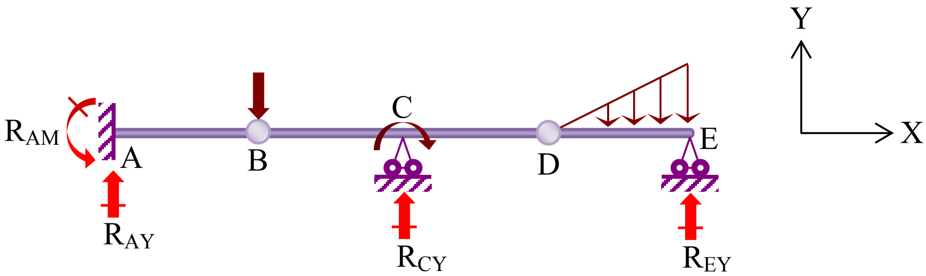


Figure 2.25: Schematic of a two-span beam containing four support reactions

To clearly demonstrate how to determine support reactions for this particular case, consider, for example, the beam shown in Figure 2.25. This structure is obviously statically determinate (i.e., r_a = 4, n_m = 2(2) = 4, n_j = 3(2) = 6, n_c = 2 → DI = 4 + 4 – 6 – 2 = 0) and this therefore ensures that all support reactions can be obtained only from static equilibrium. Since there are four unknowns reactions {R_{AM}, R_{AY}, R_{CY}, R_{EY}}, we still need to construct two additional equations, when used together with those two constructed on the entire beam, to render a sufficient number of equations. To achieve this task, let us first introduce a cut at a hinge D and consider the right part of the beam (see FBD in Figure 2.26(a)) and next introduce a cut at a point just to the right of a hinge B and consider the right part of the beam (see FBD in Figure 2.27(b)). Note that we intend not to make a cut exactly at the hinge B since we want to avoid an argument related to how to distribute a concentrated load acting at the hinge B to the left and right parts of the beam. The unknown reactions {R_{AM}, R_{AY}, R_{CY}, R_{EY}} can then be computed as follow:

- the reaction R_{EY} is obtained from equilibrium of moments about a point D of the right part of the beam shown in Figure 2.26(a),
- the reaction R_{CY} is obtained from equilibrium of moments about a point B_R of the right part of the beam shown in Figure 2.26(b),

- the reaction R_{AM} is obtained from equilibrium of moments about a point A of the entire beam shown in Figure 2.26(c), and
- the reaction R_{AY} is obtained from equilibrium of forces in Y-direction of the entire structure.

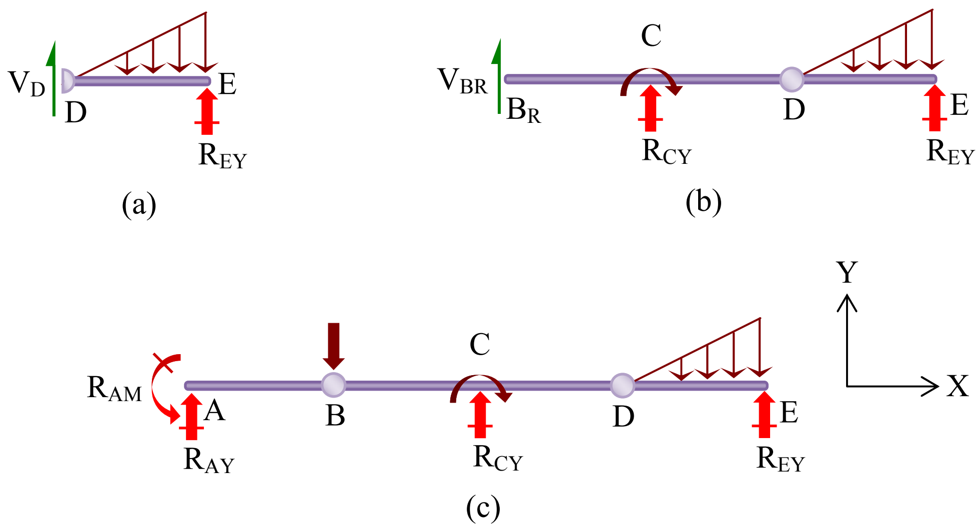


Figure 2.26: (a) FBD of right part of beam when a cut is made at hinge D, (b) FBD of right part of the beam when a cut is made at point just to the right of hinge B, and (c) FBD of entire beam

Alternatively, let us introduce two cuts simultaneously, one at point just to the right of a hinge B and the other at a hinge D. With these two cuts, we can sketch corresponding FBDs of three parts of the beam as shown in Figure 2.27. While we introduce two extra unknowns $\{V_{BR}, V_D\}$ at the cut, the total number of unknowns (4 + 2 = 6) is now equal to the number of equilibrium equations that can be set up for the three parts (2 + 2 + 2 = 6). To obtain all reactions without solving a system of six linear equations, we can consider equilibrium of each part as follow:

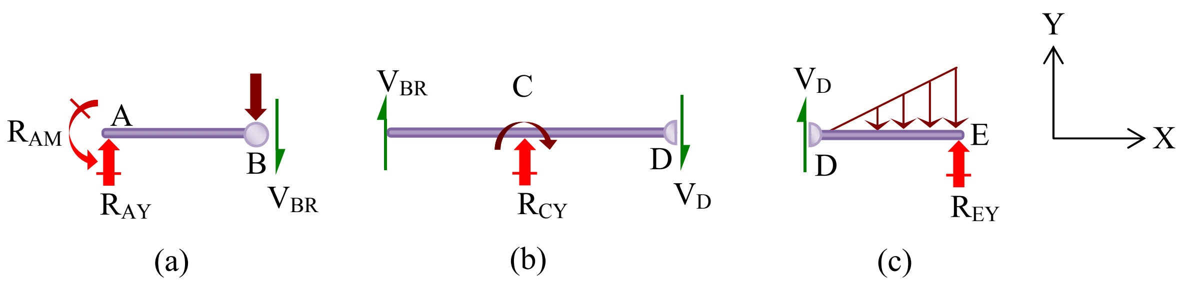


Figure 2.27: Free body diagrams of three parts of the beam resulting from two cuts at point just to the right of hinge B and at hinge D

- the reaction R_{EY} is obtained from equilibrium of moments about a point D of the right part of the beam shown in Figure 2.27(c),
- the shear force V_D is obtained from equilibrium of forces in Y-direction of the right part of the beam shown in Figure 2.27(c),
- the reaction R_{CY} is obtained from equilibrium of moments about a point B_R of the middle part of the beam shown in Figure 2.27(b),
- the shear force V_{BR} is obtained from equilibrium of forces in Y-direction of the middle part of the beam shown in Figure 2.27(b),

- the reaction R_{AM} is obtained from equilibrium of moments about a point A of the left part of the beam shown in Figure 2.27(a), and
- the reaction R_{AY} is obtained from equilibrium of forces in Y-direction of the left part of the beam shown in Figure 2.27(a).

It is remarked that while the both solution strategies yield identical results, the intermediate unknowns $\{V_{BR}, V_D\}$ must be solved in the second strategy in order to obtain all support reactions.

2.5.4 Method of sections

In various situations, the shear force and bending moment at some specific locations are of interest. The method of sections similar to that employed in the analysis of member forces in trusses can efficiently be applied here. The procedure starts with introducing a cut at a location where the shear force and bending moment are to be determined in order to access and see those unknown internal forces and then follows by applying static equilibrium equations to solve for such unknowns. If all support reactions are determined before the method of sections is applied, only two unknowns (one corresponding to the shear force and the other corresponding to the bending moment) are introduced at the cut and appear in the FBD of both parts of the beam resulting from the cut. Consideration of equilibrium of either part provides two independent equilibrium equations and this is sufficient for solving the two unknown internal forces at a particular location. Procedures for determining shear force and bending moment at a particular point P by the method of sections can be summarized as follow (see for example a beam shown in Figure 2.28):

- Determine all support reactions following guideline given in section 2.5.4
- Introduce a cut at point P and then separate the beam into two parts
- Choose one of the two parts that seems to involve less computation
- Sketch the FBD of a selected part; both shear force and bending moment are assumed a priori to be positive.
- Apply equilibrium of forces of the selected part in Y-direction; this yields the shear force at point P (V_P). If the computed shear force is positive, the assumed direction is correct; otherwise, the actual direction is opposite to the assumed direction.
- Apply equilibrium of moments of the selected part about a point P; this yields the bending moment at point P (M_P). If the computed bending moment is positive, the assumed direction is correct; otherwise, the actual direction is opposite to the assumed direction.

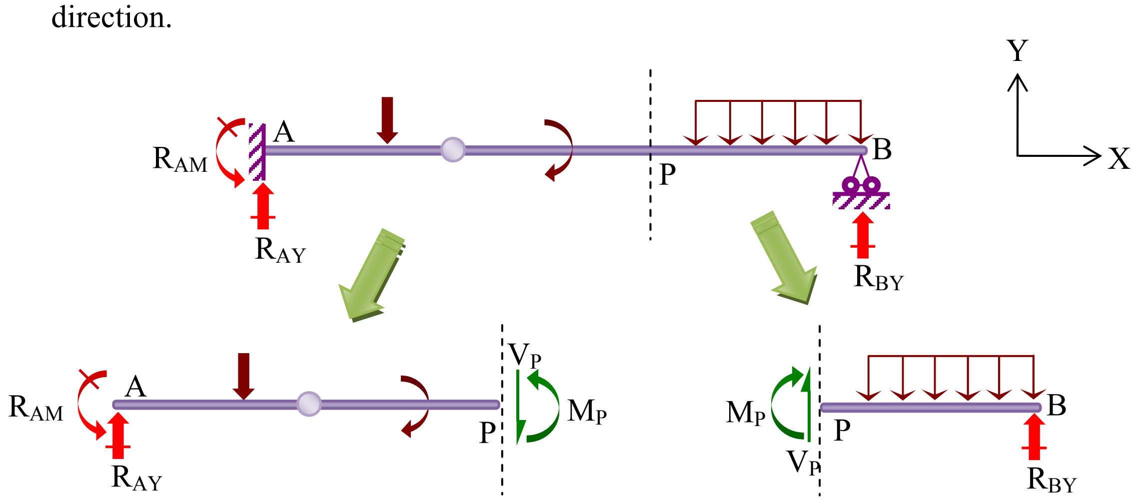
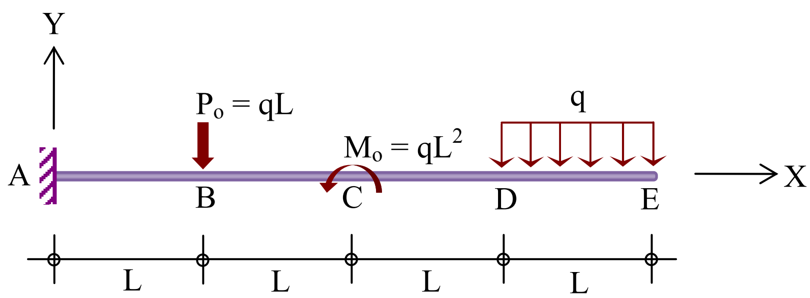


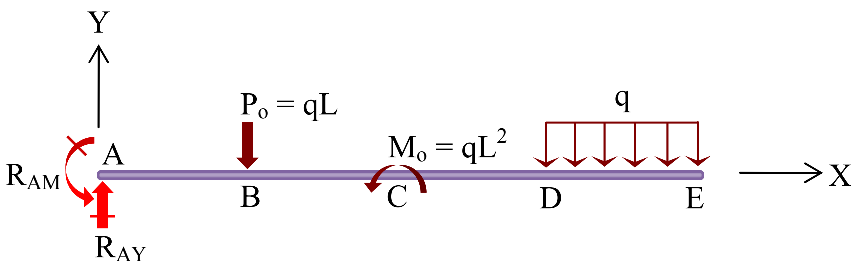
Figure 2.28: Schematic indicating a cut used to access the shear force and bending moment at point P and FBDs of the two parts resulting from the cut

It is remarked that the method of sections can be used not only to determine shear force and bending moment at a particular point but also to determine the shear force and bending moment at any location of the beam, i.e. V(x) and M(x). Procedures to obtain V(x) and M(x) are similar to those described above except that a cut must be made at any point x instead of a specific location. In addition, the entire beam needs to be separated into several subintervals due to the discontinuity induced by the presence of supports, concentrated applied loads, and locations where distributed loads change their distribution. The function forms of V(x) or M(x) for those subintervals are generally different and need to be constructed separately. Once the shear force and bending moment as a function of position x along the beam are determined, graphs of V(x) and M(x) can be plotted with the x-axis directing along the axis of the beam. These two graphical representations are known as a *shear force diagram* (SFD) and a *bending moment diagram* (BMD), respectively. Example 2.7 and Example 2.8 demonstrate applications of the method of sections to determine shear force and bending moment at some specific locations and to construct V(x) and M(x) and sketch the corresponding SFD and BMD, respectively.

Example 2.7 Determine shear force and bending moment at points A_R, B_L, B_R, C_L, C_R, D_L, D_R of a beam shown below. Subscripts L and R is used to indicate a point just to the left and a point just to the right of the indicated point, e.g. C_L, C_R are point just to the left and point just to the right of the point C.



Solution The given beam is statically determinate (i.e. $r_a = 2$, $n_m = 1(2) = 2$, $n_j = 2(2) = 4$, $n_c = 0$, then DI = 2 + 2 – 4 – 0 = 0); thus all support reactions and the internal forces at any location can be determined from static equilibrium. Since the number of support reactions is equal to 2, they can be obtained from equilibrium of the entire structure as shown below.



$[\Sigma F_Y = 0]$ ↑+ : $R_{AY} - qL - qL = 0$

$R_{AY} = 2qL$ Upward

$[\Sigma M_A = 0]$ ↺+ : $R_{Am} - (qL)(L) + qL^2 - (qL)(3L+L/2) = 0$

$R_{Am} = 7qL^2/2$ CCW

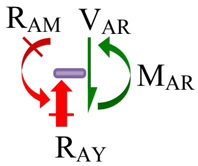
The shear force and bending moment at point A_R are obtained by making a cut at point just to the right of point A and then considering equilibrium of the left part:

$[\Sigma F_Y = 0]$ $\uparrow +$: $R_{AY} - V_{AR} = 0$

$V_{AR} = 2qL$

$[\Sigma M_A = 0]$ $\circlearrowleft +$: $R_{AM} + M_{AR} = 0$

$M_{AR} = -7qL^2/2$



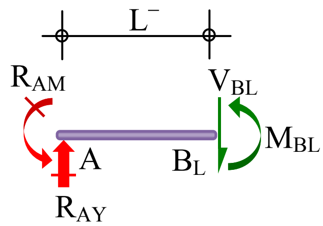
Similarly, the shear force and bending moment at point B_L are obtained by making a cut at point just to the left of point B and then considering equilibrium of the left part:

$[\Sigma F_Y = 0]$ $\uparrow +$: $R_{AY} - V_{BL} = 0$

$V_{BL} = 2qL$

$[\Sigma M_{BL} = 0]$ $\circlearrowleft +$: $R_{AM} - R_{AY}L + M_{BL} = 0$

$M_{BL} = -3qL^2/2$



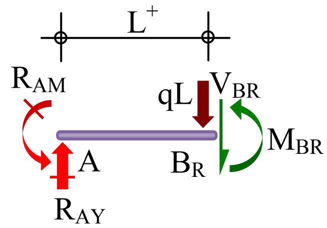
Note that the distance between the point A and point B_L is denoted by L^- and, in the limit as B_L approaches the point B, $L^- = L$. Next, the shear force and bending moment at point B_R are obtained by making a cut at point just to the right of point B and then considering equilibrium of the left part:

$[\Sigma F_Y = 0]$ $\uparrow +$: $R_{AY} - V_{BR} - qL = 0$

$V_{BR} = qL$

$[\Sigma M_{BR} = 0]$ $\circlearrowleft +$: $R_{AM} - R_{AY}L + M_{BR} = 0$

$M_{BR} = -3qL^2/2$



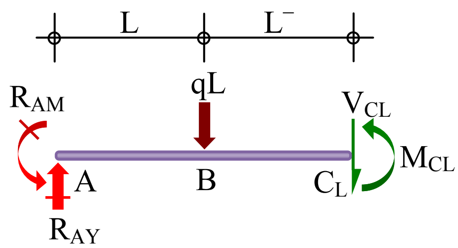
Similarly, in the limit as B_R approaches the point B, $L^+ = L$. Next, the shear force and bending moment at point C_L are obtained by making a cut at point just to the left of point C and then considering equilibrium of the left part:

$[\Sigma F_Y = 0]$ $\uparrow +$: $R_{AY} - V_{CL} - qL = 0$

$V_{CL} = qL$

$[\Sigma M_{CL} = 0]$ $\circlearrowleft +$: $R_{AM} - 2R_{AY}L + qL^2 + M_{CL} = 0$

$M_{CL} = -qL^2/2$



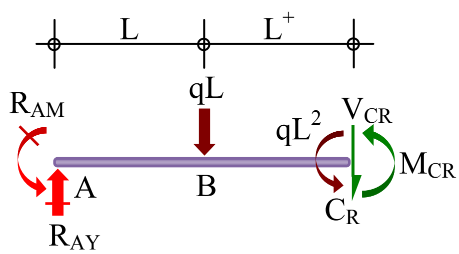
The shear force and bending moment at point C_R can be obtained by making a cut at point just to the right of point C and then considering equilibrium of the left part:

$[\Sigma F_Y = 0]$ $\uparrow +$: $R_{AY} - V_{CR} - qL = 0$

$V_{CR} = qL$

$[\Sigma M_{CR} = 0]$ $\circlearrowleft +$: $R_{AM} - 2R_{AY}L + 2qL^2 + M_{CR} = 0$

$M_{CR} = -3qL^2/2$



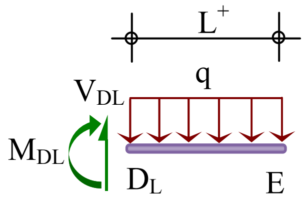
The shear force and bending moment at point D_L can be obtained by making a cut at point just to the left of point C and then considering equilibrium of the right part:

$[\Sigma F_Y = 0]$ ↑+ : $V_{DL} - qL = 0$

$V_{DL} = qL$

$[\Sigma M_{DL} = 0]$ ↺+ : $-M_{DL} - (qL)(L/2) = 0$

$M_{DL} = -qL^2/2$



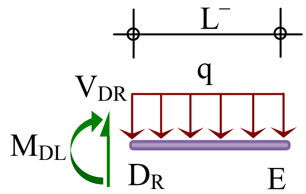
Finally, the shear force and bending moment at point D_R can be obtained by making a cut at point just to the right of point D and then considering equilibrium of the right part:

$[\Sigma F_Y = 0]$ ↑+ : $V_{DR} - qL = 0$

$V_{DR} = qL$

$[\Sigma M_{DR} = 0]$ ↺+ : $-M_{DR} - (qL)(L/2) = 0$

$M_{DR} = -qL^2/2$



From results obtained, it is worth noting that at a point where a concentrated force is applied, there is a jump of the shear force equal to the magnitude of the concentrated force while there is no jump of the bending moment. For instance, at the point B, we have

$V_{BR} - V_{BL} = -qL$

$M_{BR} - M_{BL} = 0$

Next, at a point where a concentrated moment is applied, there is a jump of the bending moment equal to the magnitude of the concentrated moment while there is no jump of the shear force. For instance, at the point C, we have

$V_{CR} - V_{CL} = 0$

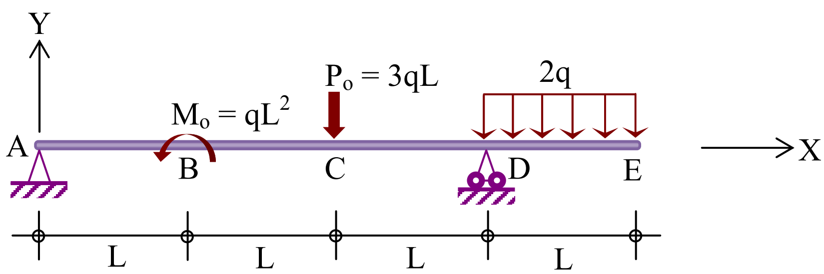
$M_{CR} - M_{CL} = -qL^2$

Finally, at a point where the distributed load is discontinuous, there is no jump of both the bending moment and the shear force. For instance, at the point D, we have

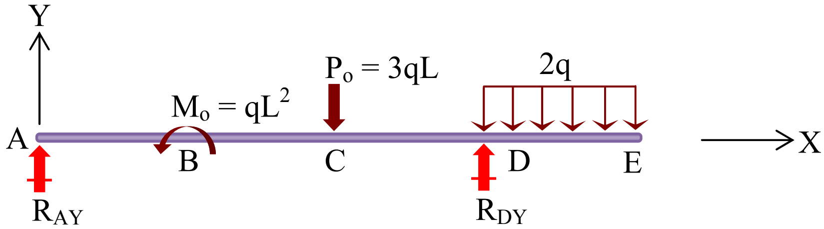
$V_{DR} - V_{DL} = 0$

$M_{DR} - M_{DL} = 0$

Example 2.8 Determine all support reactions and then use the method of sections to construct the SFD and BMD of a beam shown below.



Solution The given beam is statically determinate (i.e. $r_a = 2$, $n_m = 2(2) = 4$, $n_j = 3(2) = 6$, $n_c = 0$, then DI = 2 + 4 – 6 – 0 = 0); thus all support reactions and the internal forces at any location can be determined from static equilibrium. Since the number of support reactions is equal to 2, they can be obtained from equilibrium of the entire structure as shown below.



$[\Sigma M_A = 0]$ ↺+ : $3R_{DY}L + qL^2 - (3qL)(2L) - (2qL)(3L+L/2) = 0$

$R_{DY} = 4qL$ Upward

$[\Sigma F_Y = 0]$ ↑+ : $R_{AY} + R_{DY} - 3qL - 2qL = 0$

$R_{AY} = qL$ Upward

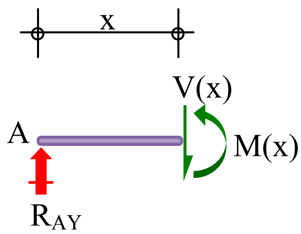
Since there are three points of discontinuity within the beam (i.e. points B, C, and D), the beam is then divided into four subintervals (i.e. subintervals AB, BC, CD and DE) and the shear force V(x) and bending moment M(x) are to be obtained for each subinterval as shown below. Starting with the subinterval AB, a cut is made at any point $x \in (0, L)$ and equilibrium of the left part of the beam is considered:

$[\Sigma F_Y = 0]$ ↑+ : $R_{AY} - V(x) = 0$

$V(x) = qL$

$[\Sigma M_x = 0]$ ↺+ : $M(x) - (R_{AY})(x) = 0$

$M(x) = qLx$



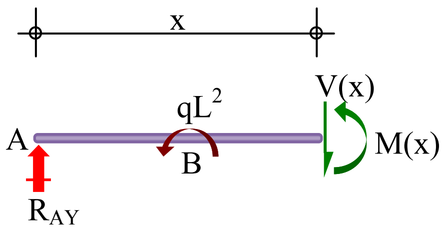
Next, the shear force V(x) and bending moment M(x) within the subinterval BC are obtained by introducing a cut at any point $x \in (L, 2L)$ and considering equilibrium of the left part of the beam:

$[\Sigma F_Y = 0]$ ↑+ : $R_{AY} - V(x) = 0$

$V(x) = qL$

$[\Sigma M_x = 0]$ ↺+ : $M(x) - (R_{AY})(x) + qL^2 = 0$

$M(x) = qLx - qL^2$



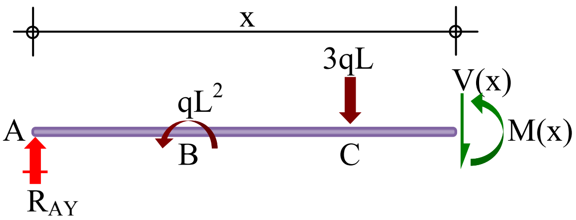
Next, the shear force V(x) and bending moment M(x) within the subinterval CD are obtained by introducing a cut at any point $x \in (2L, 3L)$ and considering equilibrium of the left part of the beam:

$[\Sigma F_Y = 0]$ ↑+ : $R_{AY} - V(x) - 3qL = 0$

$V(x) = -2qL$

$[\Sigma M_x = 0]$ ↺+ : $M(x) - (R_{AY})(x) + qL^2 + (3qL)(x\text{-}2L) = 0$

$M(x) = -2qLx + 5qL^2$



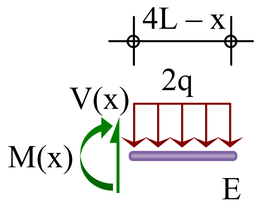
Finally, the shear force V(x) and bending moment M(x) within the last subinterval DE are obtained by making a cut at any point $x \in (3L, 4L)$ and equilibrium of the right part of the beam is considered as follows:

$[\Sigma F_Y = 0]$ ↑+ : $V(x) - 2q(4L - x) = 0$

$V(x) = 8qL - 2qx$

$[\Sigma M_x = 0]$ ↺+ : $-M(x) - (2q)(4L - x)(4L - x)/2 = 0$

$M(x) = -q(4L - x)^2$



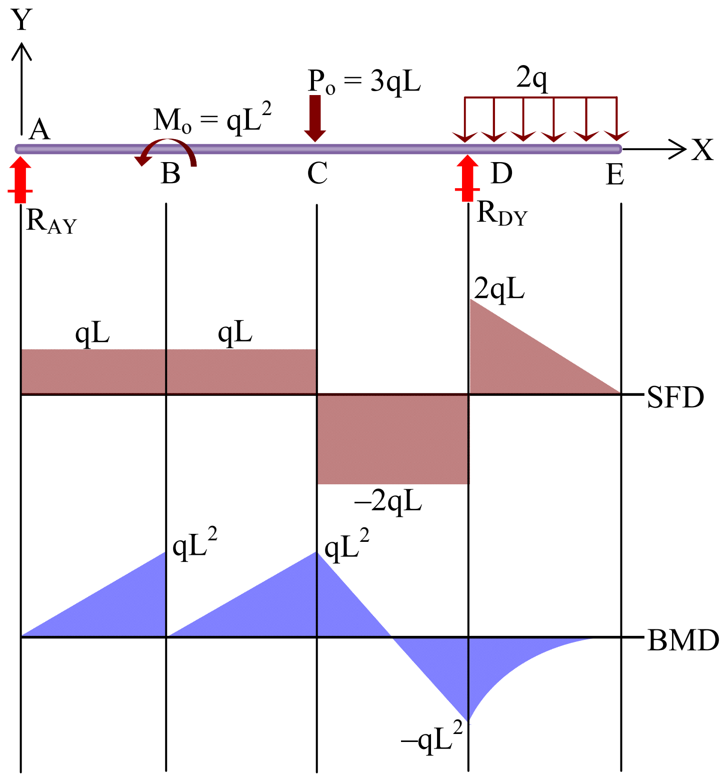
The shear force V(x) and the bending moment M(x) for the entire beam are summarized and the corresponding shear force diagram (SFD) and bending moment diagram (BMD) are sketched as shown below.

- Shear force

  $V(x) = qL$, $x \in (0, L)$
  $V(x) = qL$, $x \in (L, 2L)$
  $V(x) = -2qL$, $x \in (2L, 3L)$
  $V(x) = 8qL - 2qx$, $x \in (3L, 4L)$

- Bending moment

  $M(x) = qLx$, $x \in (0, L)$
  $M(x) = qLx - qL^2$, $x \in (L, 2L)$
  $M(x) = -2qLx + 5qL^2$, $x \in (2L, 3L)$
  $M(x) = -q(4L - x)^2$, $x \in (3L, 4L)$



From above SFD and BMD, the maximum positive shear force is equal to 2qL occurring at point just to the left of the point D; the maximum negative shear force is equal to 2qL occurring at the entire subinterval CD; the maximum positive bending moment is equal to qL^2 occurring at points B and C; and the maximum negative bending moment is equal to qL^2 occurring at point D.

2.5.5 Method of differential and integral formula

It has been apparent from the previous section that the method of sections can become inefficient when applied to construct SFD and BMD of beams. This is due to the need to obtain the shear force and bending moment as a function of position along the entire beam, i.e. V(x) and M(x); the construction of these two functions is somewhat cumbersome especially when there are many points of loading discontinuity (e.g. points where supports are present, concentrated forces and moments are applied, distributed loads changes their distribution, etc.) thus requiring to form V(x) and M(x) in several subinterval separately.

To overcome this tedious task, another technique called "*a method of differential and integral formula*" is introduced. This technique is still based primarily on static equilibrium but the key equilibrium equations employed are expressed in both differential form and integral form. The special feature of these equations is that they relate the shear force and bending moment to the external applied loads in both local and global senses and, therefore, allow the shear force and bending moment be obtained as a function of position without introducing any cut along the beam. More explanation about this technique is presented further below.

2.5.5.1 Equilibrium equations in differential form

Consider a beam that is in equilibrium with applied transverse loads as shown schematically in Figure 2.29(a). Let's introduce two cuts, one at the coordinate x and the other at the coordinate x + dx, and then sketch the FBD of an infinitesimal element dx as shown in Figure 2.29(b). The shear force and bending moment at x are denoted by V(x) and M(x), respectively, and the shear force and bending moment at x + dx are denoted by V(x) + dV and M(x) + dM, respectively, where dV and dM are increments of shear force and bending moment. Note that the positive sign convention for the shear force, bending moment, and the distributed load q (distributed load is considered to be positive if its direction is along the Y-axis) are assumed.

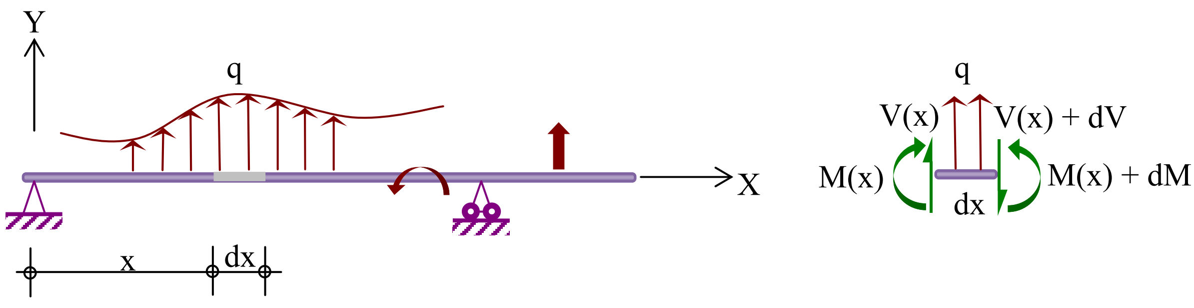


Figure 2.29: (a) Schematic of a beam subjected to transverse loads and (b) FBD of an infinitesimal element dx

By enforcing static equilibrium of the infinitesimal element dx shown in Figure 2.29(b) and then taking appropriate limit process, we obtain the following two equilibrium equations in a differential form:

$$\Sigma F_Y = 0 \quad \Rightarrow \quad V(x) + qdx - V(x) - dV = 0 \quad \Rightarrow \quad \frac{dV(x)}{dx} = q(x) \qquad (2.15)$$

$$\Sigma M_Z = 0 \quad \Rightarrow \quad M(x) + dM - M(x) - V(x)dx - qdx(dx/2) = 0 \quad \Rightarrow \quad \frac{dM(x)}{dx} = V(x) \qquad (2.16)$$

It is important to emphasize that the equation (2.15) is valid at any point x such that the distributed load q is continuous and it is free of any concentrated force while the equation (2.15) is valid at any point x such that the shear force is continuous and it is free of any concentrated moment.

The first equation (2.15) states that the spatial rate of change of the shear force or "*the slope of the shear force diagram*" at any point is equal to the value of the distributed load q applied at that point. Based on this argument, following cases can be deduced:

- A segment of the beam that is free of distributed load and concentrated force (i.e. q = 0) must have a constant shear or, equivalently, a portion of the SFD corresponding to that segment possesses a zero slope or, in the other word, assumes a horizontal straight line; for instance, segments AB, CD, FG, and IJ of a beam shown in Figure 2.30.
- The shear force of a beam segment that is subjected only to uniformly distributed load q possesses a linear distribution across that segment or, equivalently, a portion of the SFD corresponding to that beam segment assumes a straight line (e.g. segments BC and JK of a beam shown in Figure 2.30). The slope of this straight line depends on the direction of q; the slope is positive when q directs upwards otherwise it is negative.
- If the distributed load q directs upward (i.e. $q > 0$) with monotonically increasing magnitude over a segment of the beam, a portion of the SFD corresponding to that segment assumes a rising and concave upward curve (e.g. segment GH shown in Figure 2.30). On the contrary, if the magnitude of the distributed load decreases monotonically (while its direction is still upward), the corresponding portion of the SFD assumes a rising and concave downward curve (e.g. segment HI shown in Figure 2.30).
- If the distributed load q directs downward (i.e. $q < 0$) with monotonically increasing magnitude over a segment of the beam, a portion of the SFD corresponding to that segment assumes a dropping and concave downward curve (e.g. segment DE shown in Figure 2.30). On the contrary, if the magnitude of the distributed load decreases monotonically (while its direction is still downward), the corresponding portion of the SFD assumes a dropping and concave upward curve (e.g. segment EF shown in Figure 2.30).

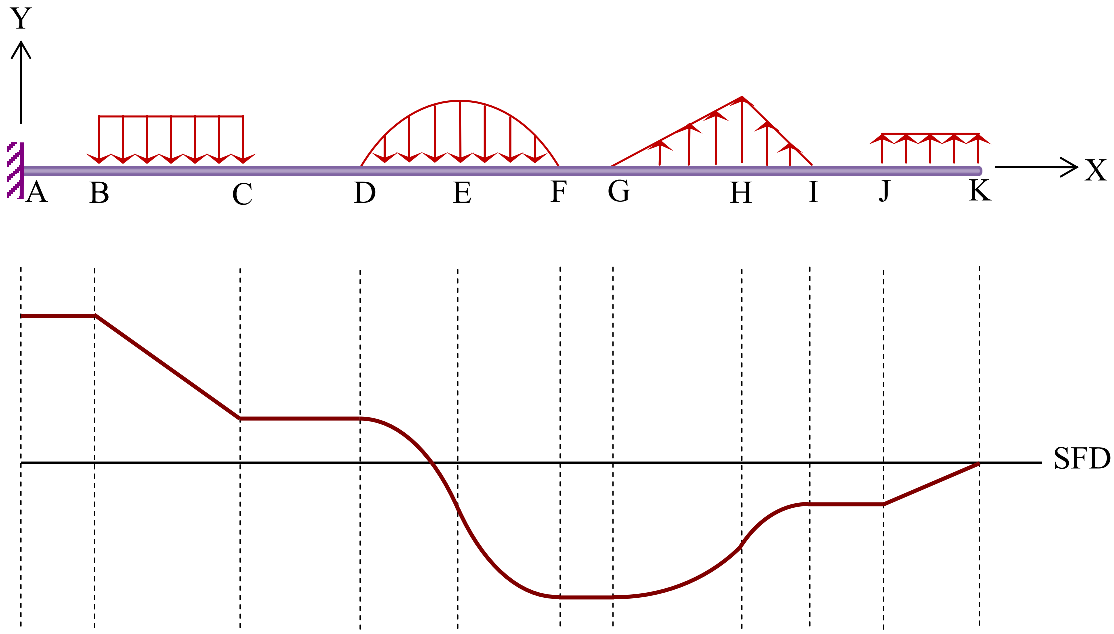


Figure 2.30: SFD of a beam subjected to different types of distributed load

The second equation (2.16) states that the spatial rate of change of the bending moment or "_the slope of the bending moment diagram_" at any point is equal to the value of shear force at that point. Based on this argument, following cases can be deduced:

- A segment of the beam that the shear force identically vanishes (i.e. V = 0) must have a constant shear or, equivalently, a portion of the BMD corresponding to that segment possesses a zero slope or, in the other word, assumes a horizontal straight line (e.g. segment AB shown in Figure 2.31).
- For a segment of the beam that possesses a constant positive shear force, the bending moment increases linearly or, equivalently, a portion of the BMD corresponding to that segment assumes a rising straight line (e.g. segment BC shown in Figure 2.31).
- For a segment of the beam that possesses a constant negative shear force, the bending moment decreases linearly or, equivalently, a portion of the BMD corresponding to that segment assumes a dropping straight line (e.g. segment GH shown in Figure 2.31).
- If the shear force is positive and increases monotonically in magnitude over a segment of the beam, a portion of the BMD corresponding to that segment assumes a rising and concave upward curve (e.g. segment EF shown in Figure 2.31).
- If the shear force is positive and decreases monotonically in magnitude over a segment of the beam, a portion of the BMD corresponding to that segment assumes a rising and concave downward curve (e.g. segment FG shown in Figure 2.31).
- If the shear force is negative and increases monotonically in magnitude over a segment of the beam, a portion of the BMD corresponding to that segment assumes a dropping and concave downward curve (e.g. segment CD shown in Figure 2.31).
- If the shear force is negative and decreases monotonically in magnitude over a segment of the beam, a portion of the BMD corresponding to that segment assumes a dropping and concave upward curve (e.g. segment DE shown in Figure 2.31).

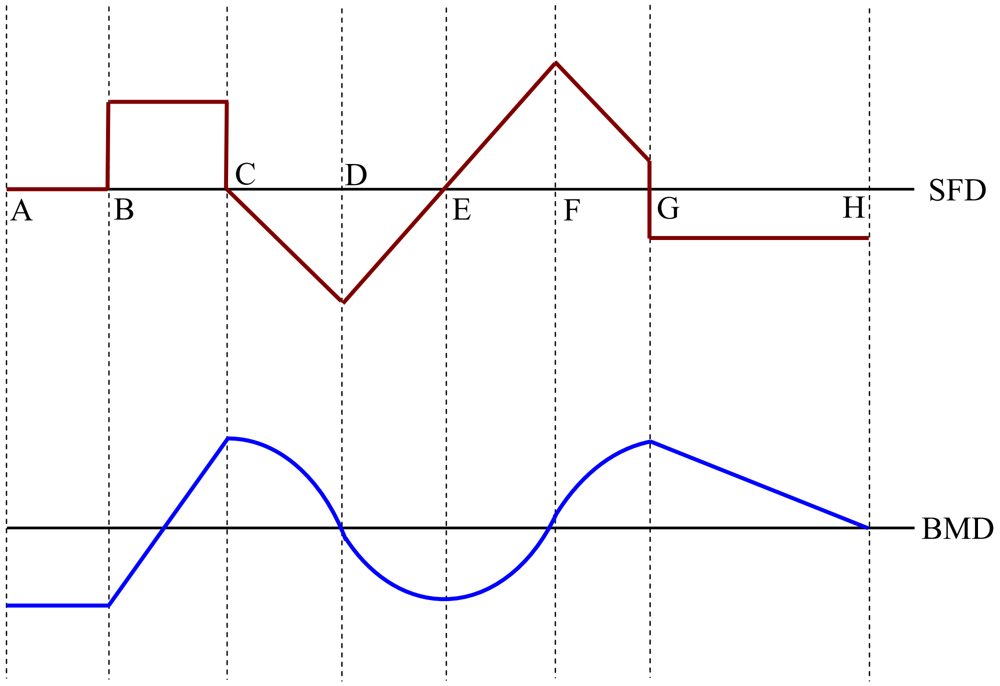


Figure 2.31: SFD and the corresponding BMD of a beam

2.5.5.2 Equilibrium equations in integral form

Now, let A and B be two points within the beam and x_A and x_B be their X-coordinates; without loss of generality, let's assume further that the point B is on the right of point A, i.e., $x_B > x_A$. By directly integrating equation (2.15) from the point A to the point B, we then obtain the integral formula

$$V_B = V_A + \int_{x_A}^{x_B} q\, dx = V_A + Q_{AB} \tag{2.17}$$

where V_A and V_B are shear forces at the points A and B, respectively, and Q_{AB} is the sum of all distributed load over the segment AB. This equation implies that the shear force at the point B can be obtained by adding the total load over the segment AB to the shear force at the point A. It is important to emphasize that equation (2.17) applies only to the case that there is no concentrated force acting to the segment AB. Note that the sign convention of the total load Q_{AB} follows that of the distributed load q.

Similarly, by directly integrating equation (2.16) from the point A to the point B, we obtain the integral formula

$$M_B = M_A + \int_{x_A}^{x_B} V\, dx = M_A + AreaV_{AB} \tag{2.18}$$

where M_A and M_B are the bending moment at the point A and point B, respectively, and $AreaV_{AB}$ is the area of the shear force diagram over the segment AB. This equation implies that the bending moment at the point B can be obtained by adding the area of the shear force diagram over the segment AB to the bending moment at the point A. It is important to emphasize that the relation (2.18) applies to the case that there is no concentrated moment acting to the portion AB. The sign convention of the area $AreaV_{AB}$ follows directly from that of the shear force V; it can therefore be either positive or negative.

Both the relations (2.17) and (2.18) can be used to determine the shear force and bending moment at a particular point when there exists at least one point that the shear force and the bending moment are known. If the relation (2.18) is to be employed in the construction of BMD, the SFD must be known a priori.

2.5.5.3 Discontinuity of shear force and bending moment

It has been found in various situations that actual applied loads are suitable to be modeled by concentrated forces or concentrated moments in the idealized structure. Presence of such concentrated loads within the beam may introduce the discontinuity of certain components of the internal force at the location where the concentrated loads are applied. Here, we employ static equilibrium to establish the discontinuity conditions of the shear force and bending moment at points where the concentrated loads are applied.

First, let us investigate the discontinuity condition at the location where a concentrate force is applied. Let P_o be a concentrated force applied to a point A of the beam (this force is considered to be positive if it directs upward in Y-direction otherwise it is negative). By introducing two cuts at a point just to left and a point just to the right of the point A and then considering equilibrium of an infinitesimal element containing a point A (its free body diagram is shown in Figure 2.32) along with taking appropriate limit process, we obtain the discontinuity conditions of the shear force and bending moment:

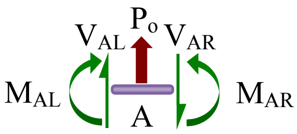


Figure 2.32: FBD of infinitesimal element containing a point where concentrated force is applied

$$V_{AR} = V_{AL} + P_o \tag{2.19}$$

$$M_{AR} = M_{AL} \tag{2.20}$$

where V_{AR} and V_{AL} are the shear force at a point just to the right and a point just to the left of the point A, respectively, and M_{AR} and M_{AL} are the bending moment at a point just to the right and a point just to the left of the point A, respectively. It is evident from (2.19) and (2.20) that, at the location where a concentrated force is applied, the shear force is not defined and is discontinuous with the magnitude of the jump equal to the magnitude of the concentrated force while the bending moment is still continuous. In addition, the shear force experiences a positive jump if the concentrated force is positive (or directs upward in the Y-direction) and it experiences a negative jump if the concentrated force is negative (or directs downward in the opposite Y-direction).

Next, let's consider the discontinuity condition at a location where a concentrated moment M_o is applied. Let M_o be a concentrated moment applied to a point A (this moment is considered to be positive if it directs in a counter clockwise direction or in the Z-direction otherwise it is negative). By introducing two cuts at a point just to left and a point just to the right of the point A and then considering equilibrium of an infinitesimal element containing a point A (its free body diagram is shown in Figure 2.33) along with taking appropriate limit process, we obtain the discontinuity conditions of the shear force and bending moment:

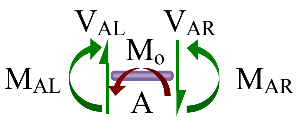


Figure 2.33: FBD of infinitesimal element containing a point where concentrated moment is applied

$$V_{AR} = V_{AL} \tag{2.21}$$

$$M_{AR} = M_{AL} - M_o \tag{2.22}$$

Equations (2.21) and (2.22) imply that at a point where the concentrated moment is applied, the bending moment is discontinuous with the magnitude of the jump equal to the magnitude of the concentrated moment while the shear force experiences no jump but it is not defined at this point. We emphasize in addition that the bending moment experiences a positive jump if the concentrated moment is negative (or directs in a clockwise direction or Z-direction) and it experiences a negative jump if the concentrated moment is positive (or directs in a counter clockwise direction or opposite Z-direction).

Finally, let us investigate the location where the distributed load q is discontinuous, say a point A. By introducing two cuts at a point just to left and a point just to the right of the point A and then considering equilibrium of an infinitesimal element containing a point A (its free body diagram is shown in Figure 2.34) along with taking appropriate limit process, we obtain the discontinuity conditions of the shear force and bending moment:

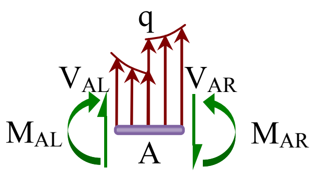


Figure 2.34: FBD of infinitesimal element containing point where distributed load is discontinuous

$$V_{AR} = V_{AL} \tag{2.23}$$

$$M_{AR} = M_{AL} \tag{2.24}$$

This implies that both the shear force and bending moment are continuous at a location where the distributed load is discontinuous.

2.5.5.4 Procedures for constructing SFD and BMD

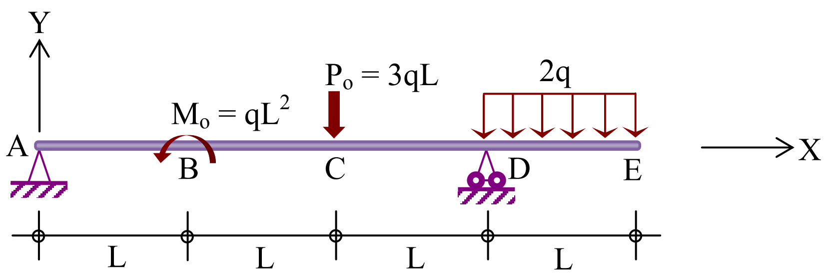
The differential formula (2.15) and (2.16), the integral formula (2.17) and (2.18) and the discontinuity conditions (2.19)-(2.24) constitutes basic components for constructing the SFD and BMD of a beam. In particular, the two integral formula (2.17) and (2.18) are employed to obtain the shear force and the bending moment at the right end of any segment when values at the left end of those quantities are known and there is no point of loading discontinuity within the segment (e.g. concentrated forces and moments). The two differential formulae (2.15) and (2.16) are then used to identify the type of a curve that connects a part of the SFD and BMD over a segment where values of the shear force and bending moment are already known at its ends. The discontinuity conditions are used to dictate the jump of the shear force and bending moment in the SFD and BMD due to the presence of concentrated forces and moments. Here, we summarize standard procedures or guidelines for constructing the SFD and BMD of a beam.

- Determine all support reactions
- Identify and mark points of loading discontinuity, e.g. supports, points where concentrated forces and moments are applied, and points where distributed load changes its distribution
- Identify all possible segments such that points of loading discontinuity must be at the ends of each segment
- Identify a point that both the shear force and bending moment are known as a starting point (in general, <u>the left end of the beam is chosen</u> since all forces and moments are known at both ends of the beam once the reactions are already determined.)
- Start drawing the SFD as follow: (i) start with the first segment on the left of the beam since the shear force at the left end of this segment is already known, (ii) use the integral formula (2.17) to compute the shear force at the right end of the segment, (iii) use the differential formula (2.15) to identify the type of a curve and then use that curve to draw the SFD over the segment, (iv) choose the segment just to the left of the previous segment as a current segment to be considered, and (v) use the jump conditions (2.19), (2.21) and (2.23) along with the value of the shear force at the right end of the previous segment to obtain the shear force at left end of the current segment. Repeat from steps (ii) until all segments are considered. Note that, for the last segment, the shear force at the right end of the segment must be consistent with the force applied at that end.
- Once the SFD is obtained, the BMD can be constructed as follow: (i) start with the first segment on the left of the beam since the bending moment at the left end of this segment

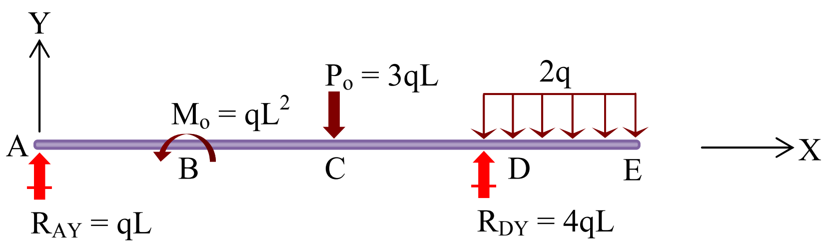
is already known, (ii) use the integral formula (2.18) to compute the bending moment at the right end of the segment, (iii) use the differential formula (2.16) to identify the type of a curve and then use that curve to draw the BMD over the segment, (iv) choose the segment just to the left of the previous segment as a current segment to be considered, and (v) use the jump conditions (2.20), (2.22) and (2.24) along with the value of the bending moment at the right end of the previous segment to obtain the bending moment at the left end of the current segment. Repeat from steps 2.2 until all segments are considered. Note that, for the last segment, the bending moment at the right end of the segment must be consistent with the moment applied at that end.

Once the SFD and BMD are completed, one can identify both the magnitude and location of the maximum shear force and maximum bending moment. In general, the maximum shear force can occur at the supports, the locations where the distributed load q vanishes, and locations where the concentrated forces are applied. Similarly, the maximum bending moment can occur at the supports, the locations where the shear force vanishes, the locations where the shear force changes its sign, and the locations where the concentrated moments are applied.

Example 2.9 Draw the SFD and BMD of a beam shown in the example 2.8 using the method of differential and integral formula



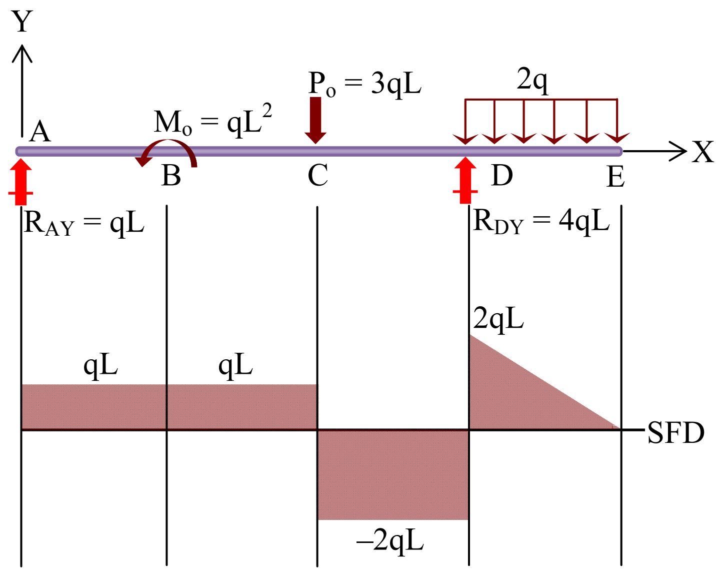
Solution All support reactions of this beam were already determined in the example 2.8 and the FBD diagram of the entire beam is shown again below.



From above FBD, there are three points of loading discontinuity within the beam (excluding the two ends of the beam), e.g. points B, C and D. Specifically, the point B is a point where the concentrated moment is applied, the point C is a point where the concentrated force is applied, and the point D is a point where the support reaction (viewed as the concentrated force) is applied and the distributed load is discontinuous. Thus, we can divide the entire beam into four segments: AB, BC, CD and DE. The shear force and the bending moment at the left end of the beam (i.e. point A) are already known, i.e. $V_A = R_{AY} = qL$ and $M_A = 0$.

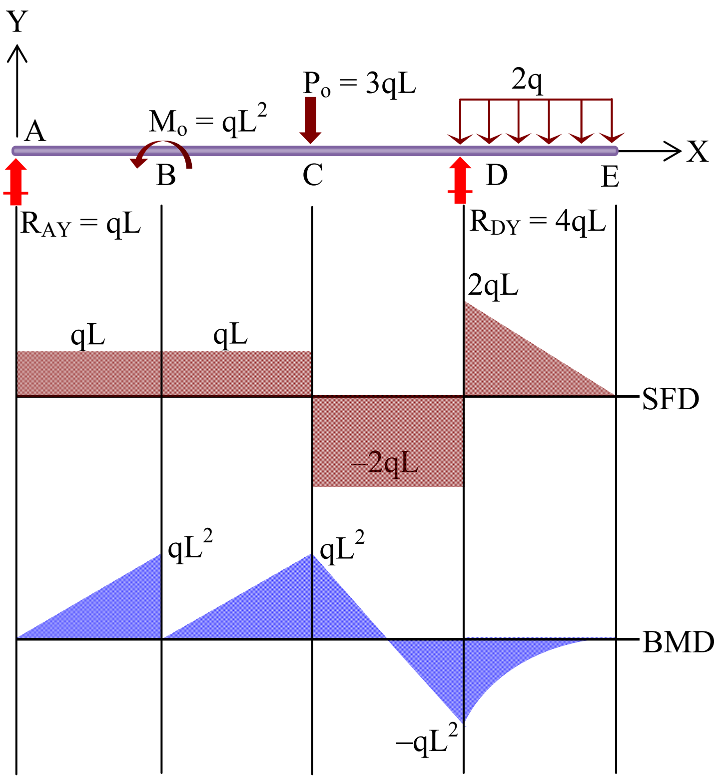
First, let us construct the SFD. The differential formula (2.15), the integral formula (2.17), and the discontinuity conditions (2.19), (2.21) and (2.23) are utilized for each segment as follow:

- Segment AB
  - $V_A = qL$
  - There is no distributed load over the segment + equation (2.17) $\Rightarrow$ $V_{BL} = V_A + 0 = qL$
  - There is no distributed load over the segment + equation (2.15) $\Rightarrow$ SFD over the segment is a horizontal straight line
- Segment BC
  - The concentrated moment is applied at point B + equation (2.21) $\Rightarrow$ there is no jump of the shear force at point B $\Rightarrow$ $V_{BR} = V_{BL} = qL$
  - There is no distributed load over the segment + equation (2.17) $\Rightarrow$ $V_{CL} = V_{BR} + 0 = qL$
  - There is no distributed load over the segment + equation (2.15) $\Rightarrow$ SFD over the segment is a horizontal straight line
- Segment CD
  - The negative concentrated force equal to 3qL is applied at point C + equation (2.19) $\Rightarrow$ there is a jump of the shear force at point C $\Rightarrow$ $V_{CR} = V_{CL} - 3qL = -2qL$
  - There is no distributed load over the segment + equation (2.17) $\Rightarrow$ $V_{DL} = V_{CR} + 0 = -2qL$
  - There is no distributed load over the segment + equation (2.15) $\Rightarrow$ SFD over the segment is a horizontal straight line
- Segment DE
  - The positive concentrated force equal to 4qL is applied at point D $\Rightarrow$ there is a jump of the shear force at point D $\Rightarrow$ $V_{DR} = V_{DL} + 4qL = 2qL$
  - A negative uniform distributed load –2q is applied over the segment + equation (2.17) $\Rightarrow$ $V_E = V_{DR} + (-2q)(L) = 0$ $\Rightarrow$ consistent with condition at the right end of the beam
  - A negative uniform distributed load –2q is applied over the segment + equation (2.15) $\Rightarrow$ SFD over the segment is a dropping straight line

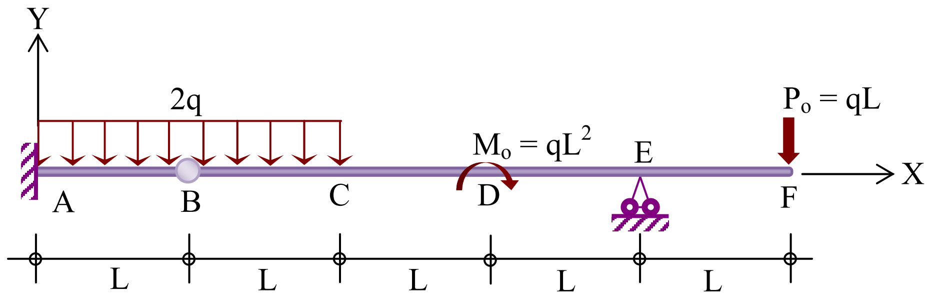


Once the SFD is obtained, the BMD can then be constructed. The differential formula (2.16), the integral formula (2.18), and the discontinuity conditions (2.20), (2.22) and (2.24) are utilized for each segment as follow:

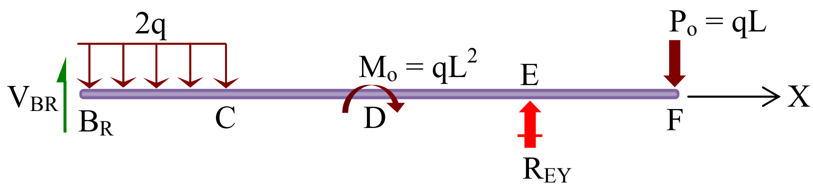
- Segment AB
  - $M_A = 0$
  - Area of the SFD over the segment is (qL)(L) + equation (2.18) $\Rightarrow M_{BL} = M_A + (qL)(L) = qL^2$
  - The shear force is constant and positive over the segment + equation (2.16) $\Rightarrow$ BMD over the segment is a rising straight line
- Segment BC
  - The positive concentrated moment equal to qL^2 is applied at point B + equation (2.22) $\Rightarrow$ there is a jump of the bending moment at point B $\Rightarrow M_{BR} = M_{BL} - qL^2 = 0$
  - Area of the SFD over the segment is (qL)(L) + equation (2.18) $\Rightarrow M_{CL} = M_{BR} + (qL)(L) = qL^2$
  - The shear force is constant and positive over the segment + equation (2.16) $\Rightarrow$ BMD over the segment is a rising straight line
- Segment CD
  - The concentrated force is applied at point C + equation (2.20) $\Rightarrow$ there is no discontinuity of the bending moment at point C $\Rightarrow M_{CR} = M_{CL} = qL^2$
  - Area of the SFD over the segment is (–2qL)(L) + equation (2.18) $\Rightarrow M_{DL} = M_{CR} + (-2qL)(L) = -qL^2$
  - The shear force is constant and negative over the segment + equation (2.16) $\Rightarrow$ BMD over the segment is a dropping straight line
- Segment DE
  - The concentrated force is applied at point D + equation (2.20) $\Rightarrow$ there is no discontinuity of the bending moment at point D $\Rightarrow M_{DR} = M_{DL} = -qL^2$
  - Area of the SFD over the segment is (2qL)(L)/2 + equation (2.18) $\Rightarrow M_{EL} = M_{DR} + (2qL)(L)/2 = 0 \Rightarrow$ consistent with condition at the right end of the beam
  - The shear force is positive and decreases monotonically in magnitude over a segment + equation (2.16) $\Rightarrow$ BMD over the segment is a rising and concave upward curve



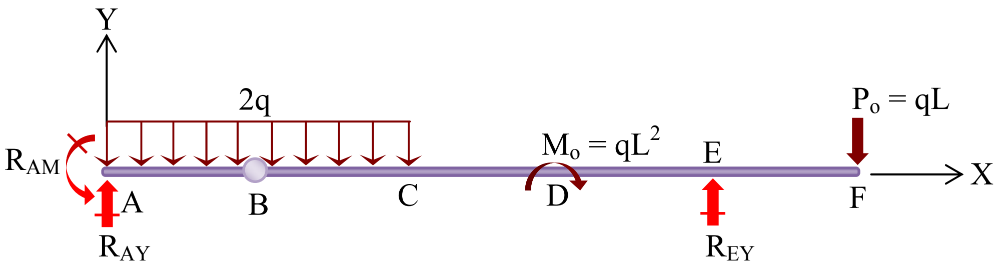
Example 2.10 Draw the SFD and BMD of a beam shown below using the method of differential and integral formula.



Solution The given beam is statically determinate (i.e. $r_a = 2 + 1 = 3$, $n_m = 2(2) = 4$, $n_j = 3(2) = 6$, $n_c = 1$, then DI = 3 + 4 – 6 – 1 = 0); thus, all support reactions and the internal forces at any location can be determined from static equilibrium. Since the number of support reactions is equal to 3, they cannot be obtained from equilibrium of the entire structure alone. By making a cut at point just to the left of the hinge B and then considering equilibrium of the right part (part BF), one of the reactions can be determined. The rest of the reactions can be computed from equilibrium of the entire beam. Details of calculation are shown below:



FBD of part BF



FBD of entire beam

Equilibrium of part BF:

$[\Sigma M_{BR} = 0]$ ↺+ : $3R_{EY}L - (2q)(L)(L/2) - qL^2 - (qL)(4L) = 0$

$R_{EY} = 2qL$ Upward

Equilibrium of entire beam:

$[\Sigma M_A = 0]$ ↺+ : $R_{AM} + 4R_{EY}L - (2q)(2L)(L) - qL^2 - (qL)(5L) = 0$

$R_{AM} = 2qL^2$ CCW

$[\Sigma F_Y = 0]$ ↑+ : $R_{AY} + R_{EY} - (2q)(2L) - qL = 0$

$R_{AY} = 3qL$ Upward

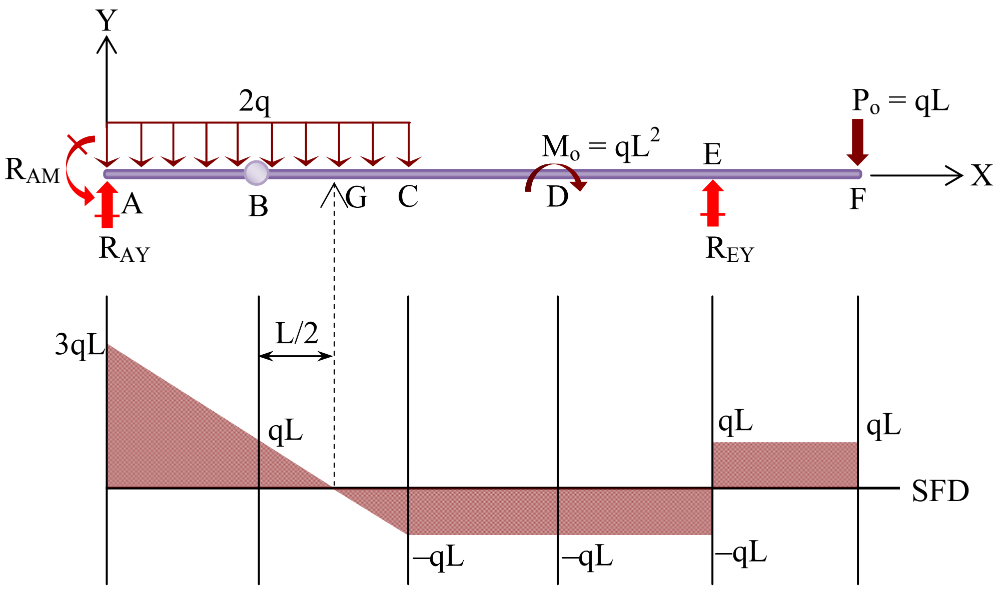
From the FBD of the entire structure, there are three points of loading discontinuity within the beam (excluding the two ends of the beam), e.g. points C, D and E. Specifically, the point C is a point where the distributed load is discontinuous, the point D is a point where the concentrated moment is applied, and the point E is a point where the support reaction (viewed as the concentrated force) is applied. Note that the point B, while it is an internal release, is not considered as a point of loading discontinuity since, at this point, the distributed load is continuous and there is no applied concentrated load. Therefore, we can divide the entire beam into four segments: AC, CD, DE and EF. Once the support reactions are determined, the shear force and the bending moment at the left end of the beam (i.e. point A) are known, i.e. $V_A = R_{AY} = 3qL$ and $M_A = –R_{AM} = –2qL^2$.

First, let us construct the SFD. The differential formula (2.15), the integral formula (2.17), and the discontinuity conditions (2.19), (2.21) and (2.23) are utilized for each segment as described below:

- Segment AC
  - $V_A = 3qL$
  - A negative uniform distributed load –2q is applied over the segment + equation (2.17) $\Rightarrow$ $V_{CL} = V_A + (–2q)(2L) = –qL$
  - A negative uniform distributed load –2q is applied over the segment + equation (2.15) $\Rightarrow$ SFD over the segment is a dropping straight line
- Segment CD
  - The distributed load is discontinuous at point C + equation (2.23) $\Rightarrow$ there is no discontinuity of the shear force at point C $\Rightarrow$ $V_{CR} = V_{CL} = –qL$
  - There is no distributed load over the segment + equation (2.17) $\Rightarrow$ $V_{DL} = V_{CR} + 0 = –qL$
  - There is no distributed load over the segment + equation (2.15) $\Rightarrow$ SFD over the segment is a horizontal straight line
- Segment DE
  - The negative concentrated moment is applied at point C + equation (2.21) $\Rightarrow$ there is no discontinuity of the shear force at point D $\Rightarrow$ $V_{DR} = V_{DL} = –qL$
  - There is no distributed load over the segment + equation (2.17) $\Rightarrow$ $V_{EL} = V_{DR} + 0 = –qL$
  - There is no distributed load over the segment + equation (2.15) $\Rightarrow$ SFD over the segment is a horizontal straight line
- Segment EF
  - The positive concentrated force equal to 2qL is applied at point E + equation (2.19) $\Rightarrow$ there is a jump of the shear force at point E $\Rightarrow$ $V_{ER} = V_{EL} + 2qL = qL$
  - There is no distributed load over the segment + equation (2.17) $\Rightarrow$ $V_{FL} = V_{ER} + 0 = qL$ $\Rightarrow$ consistent with condition at the right end of the beam
  - There is no distributed load over the segment + equation (2.15) $\Rightarrow$ SFD over the segment is a horizontal straight line

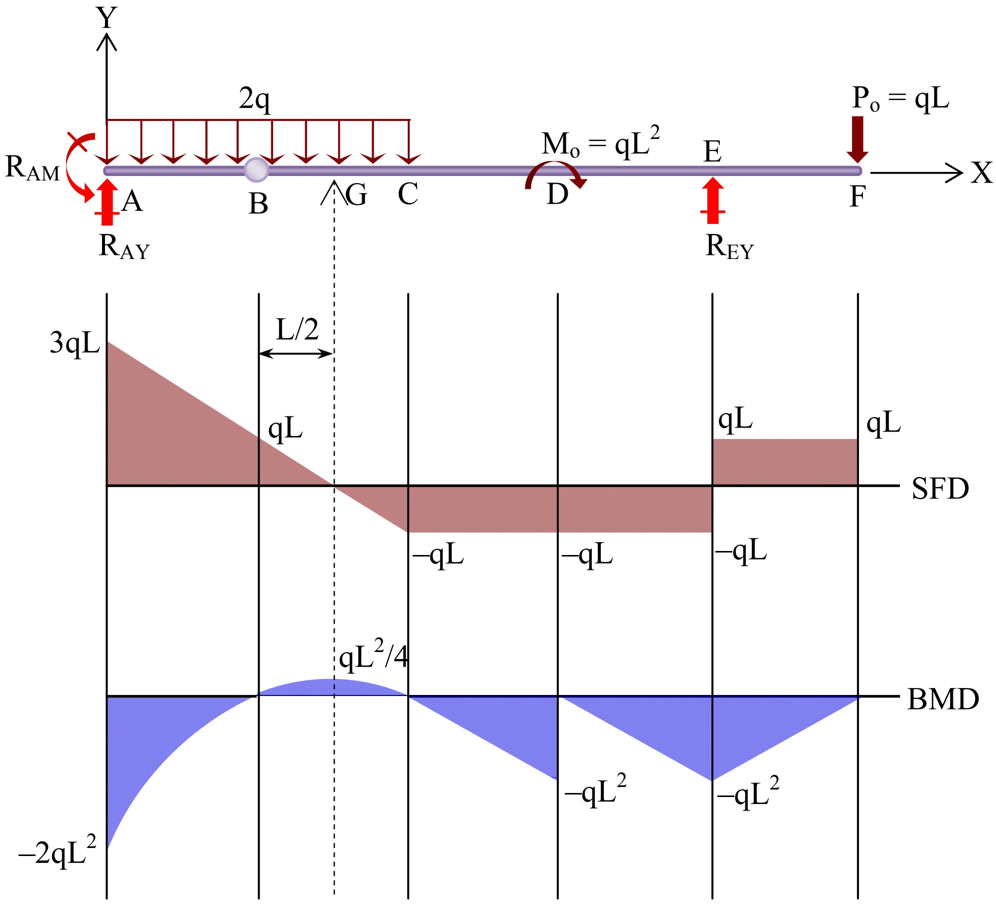
The SFD is shown in the figure below. It is evident that there exists a point within the segment AC, say a point G, such that the shear force vanishes. In particular, the point G is located, with a distance L/2, on the right of the hinge B. The segment AC can further be separated into two sub-segments AG and GC; the shear force is positive for the first sub-segment while it is negative for the second sub-segment.

In the construction of the BMD, the differential formula (2.16), the integral formula (2.18), and the discontinuity conditions (2.20), (2.22) and (2.24) are utilized for each segment as described below:



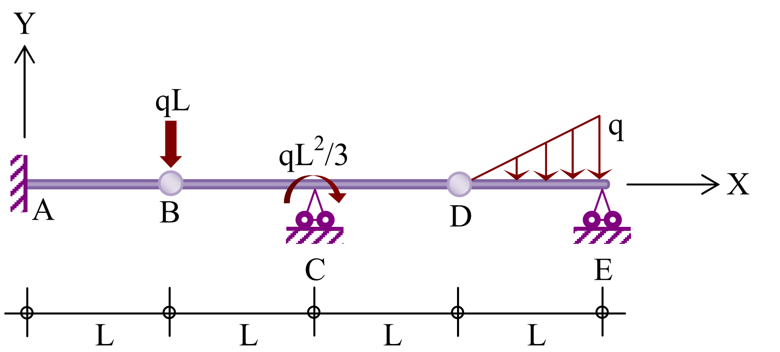
- Segment AG
  - $M_A = -R_{AM} = -2qL^2$
  - Area of the SFD over the segment is (3qL)(3L/2)/2 + equation (2.18) ⇒ $M_{GL} = M_A + (3qL)(3L/2)/2 = qL^2/4$
  - The shear force is positive and decreases monotonically in magnitude over a segment + equation (2.16) ⇒ BMD over the segment is a rising and concave downward curve with a zero slope at point G
- Segment GC
  - No loading discontinuity at point G ⇒ there is no discontinuity of the bending moment at point G ⇒ $M_{GR} = M_{GL} = 0$
  - Area of the SFD over the segment is (–qL)(L/2)/2 + equation (2.18) ⇒ $M_{CL} = M_{GR} + (-qL)(L/2)/2 = 0$
  - The shear force is negative and increases monotonically in magnitude over a segment + equation (2.16) ⇒ BMD over the segment is a dropping and concave downward curve
- Segment CD
  - The distributed load is discontinuous at point C + equation (2.24) ⇒ there is no discontinuity of the bending moment at point C ⇒ $M_{CR} = M_{CL} = 0$
  - Area of the SFD over the segment is (–qL)(L) + equation (2.18) ⇒ $M_{DL} = M_{CR} + (-qL)(L) = -qL^2$
  - The shear force is constant and negative over the segment + equation (2.16) ⇒ BMD over the segment is a dropping straight line
- Segment DE
  - The negative concentrated moment $-qL^2$ is applied at point D + equation (2.22) ⇒ there is a jump of the bending moment at point D ⇒ $M_{DR} = M_{DL} - (-qL^2) = 0$
  - Area of the SFD over the segment is (–qL)(L) + equation (2.18) ⇒ $M_{EL} = M_{DR} + (-qL)(L) = -qL^2$
  - The shear force is constant and negative over the segment + equation (2.16) ⇒ BMD over the segment is a dropping straight line
- Segment EF
  - The positive concentrated force 2qL is applied at point E + equation (2.20) ⇒ there is no discontinuity of the bending moment at point E ⇒ $M_{ER} = M_{EL} = -qL^2$

- Area of the SFD over the segment is (–qL)(L) + equation (2.18) ⇒ $M_{FL} = M_{ER}$ + (qL)(L) = 0 ⇒ consistent with the condition at the right end of the beam
- The shear force is constant and positive over the segment + equation (2.16) ⇒ BMD over the segment is a rising straight line



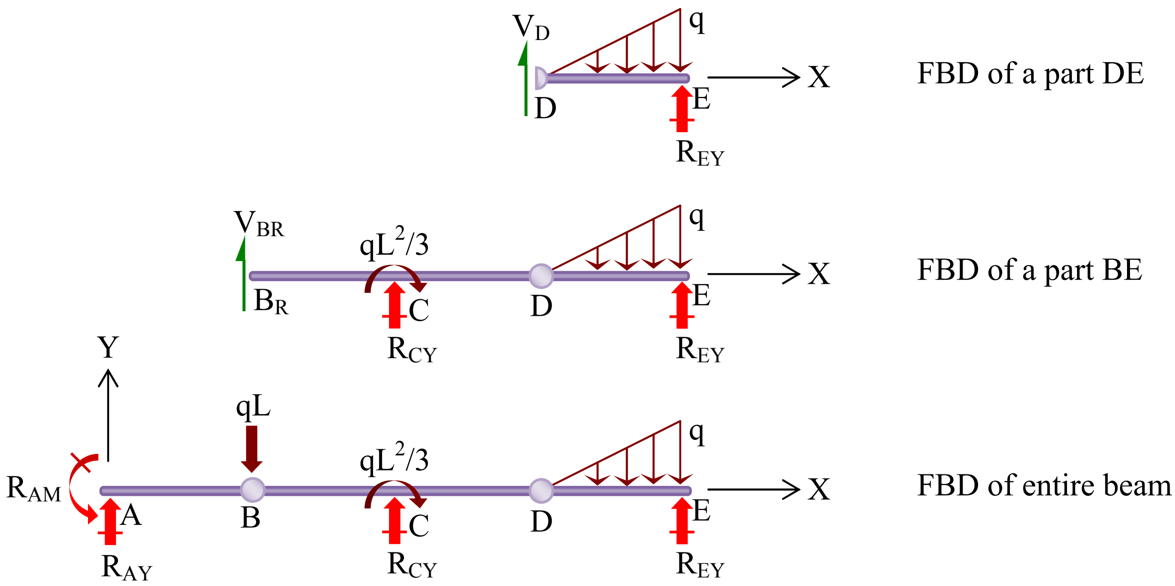
From above SFD and BMD, the maximum positive shear force is equal to 3qL occurring at point A; the maximum negative shear force is equal to qL occurring at the entire subinterval CE; the maximum positive bending moment is equal to $qL^2/4$ occurring at point G; and the maximum negative bending moment is equal to $2qL^2$ occurring at point A.

Example 2.11 Draw the SFD and BMD of a beam shown below using the method of differential and integral formula



Solution Since $r_a = 2 + 1 + 1 = 4$, $n_m = 2(2) = 4$, $n_j = 3(2) = 6$, $n_c = 2$, then DI = 4 + 4 – 6 – 2 = 0. Thus, the structure is statically determinate and all support reactions can be determined form static equilibrium. However, the number of independent equilibrium equations that can be set up for beams is $n_{et} = 2 < r_a$; thus, the support reactions cannot be obtained by considering only equilibrium of the entire structure. To overcome this problem, two additional equations associated with the presence of two moment releases or hinges at points B and D, i.e. M = 0 at B and M = 0 at D, must be employed.

By introducing a cut at the point D and considering moment equilibrium of the right part of the beam, the reaction R_{EY} can be determined; by introducing a cut at the point just to the right of the point B and considering moment equilibrium of the right part of the beam, the reaction R_{CY} can be determined; finally, by considering equilibrium of the entire beam, the rest of reactions can readily be determined. Details of calculations are shown below:



Equilibrium of part DE

$[\Sigma M_{DR} = 0]$ ↺+ : $R_{EY}L - (q)(L/2)(2L/3) = 0$

$R_{EY} = qL/3$ Upward

Equilibrium of part BE

$[\Sigma M_{DR} = 0]$ ↺+ : $R_{CY}L + 3R_{EY}L - qL^2/3 - (q)(L/2)(2L+2L/3) = 0$

$R_{CY} = 2qL/3$ Upward

Equilibrium of entire beam

$[\Sigma M_A = 0]$ ↺+ : $R_{AM} + 2R_{CY}L + 4R_{EY}L - (qL)(L) - qL^2/3 - (qL/2)(3L+2L/3) = 0$

$R_{AM} = qL^2/2$ CCW

$[\Sigma F_Y = 0]$ ↑+ : $R_{AY} + R_{CY} + R_{EY} - qL - (q)(L/2) = 0$

$R_{AY} = qL/2$ Upward

From the FBD of the entire structure, there are three points of loading discontinuity within the beam (excluding the two ends of the beam), e.g. points B, C and D. Specifically, the point B is a point where the concentrated force is applied, the point C is a point where the concentrated moment and the support reaction (viewed as the concentrated force) are applied, and the point D is a point where the distributed load changes its distribution but still continuous. Therefore, we can divide the entire beam into four segments: AB, BC, CD and DE. Once the support reactions are determined, the shear force and the bending moment at the left end of the beam (i.e. point A) are known, i.e. $V_A = R_{AY} = qL/2$ and $M_A = -R_{AM} = -qL^2/2$.

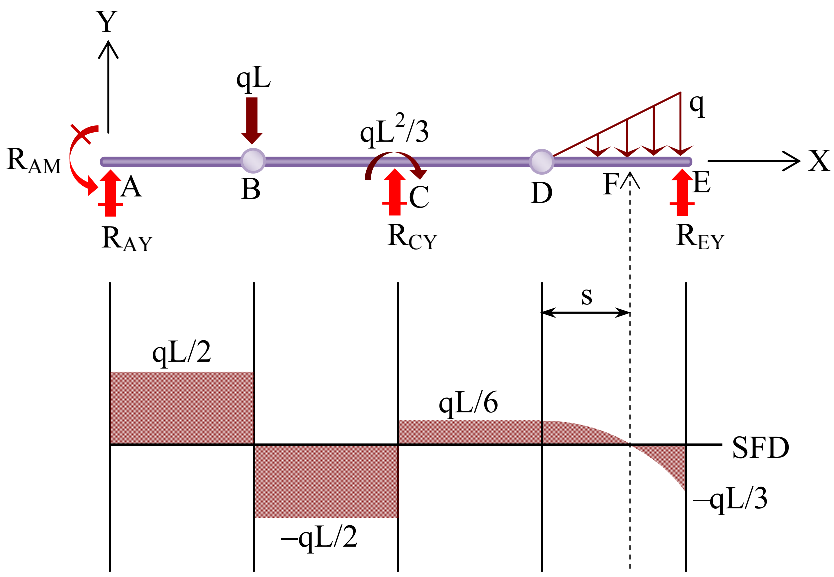
First, let us construct the SFD. The differential formula (2.15), the integral formula (2.17), and the discontinuity conditions (2.19), (2.21) and (2.23) are utilized for each segment as described below:

- Segment AB
  - $V_A = qL/2$
  - There is no distributed load over the segment + equation (2.17) $\Rightarrow$ $V_{BL} = V_A + 0 = qL/2$
  - There is no distributed load over the segment + equation (2.15) $\Rightarrow$ SFD over the segment is a horizontal straight line
- Segment BC
  - The negative concentrated force $-qL$ is applied at point B + equation (2.19) $\Rightarrow$ there is a jump of the shear force at point B $\Rightarrow$ $V_{BR} = V_{BL} + (-qL) = -qL/2$
  - There is no distributed load over the segment + equation (2.17) $\Rightarrow$ $V_{CL} = V_{BR} + 0 = -qL/2$
  - There is no distributed load over the segment + equation (2.15) $\Rightarrow$ SFD over the segment is a horizontal straight line
- Segment CD
  - The positive concentrated force $2qL/3$ is applied at point C + equation (2.19) $\Rightarrow$ there is a jump of the shear force at point C $\Rightarrow$ $V_{CR} = V_{CL} + (2qL/3) = qL/6$
  - There is no distributed load over the segment + equation (2.17) $\Rightarrow$ $V_{DL} = V_{CR} + 0 = qL/6$
  - There is no distributed load over the segment + equation (2.15) $\Rightarrow$ SFD over the segment is a horizontal straight line
- Segment DE
  - The distributed load changes its distribution at point D $\Rightarrow$ there is no discontinuity of the shear force at point D $\Rightarrow$ $V_{DR} = V_{DL} = qL/6$
  - The distributed load is negative and increases monotonically in magnitude over the segment + equation (2.17) $\Rightarrow$ $V_{EL} = V_{DR} - (q)(L)/2 = -qL/3$ $\Rightarrow$ consistent with the condition at the right end of the beam
  - The distributed load is negative and increases monotonically in magnitude over the segment + equation (2.15) $\Rightarrow$ SFD over the segment is a dropping and concave downward curve

From the SFD shown above, it is evident that there exists a point within the segment DE, say a point F, such that the shear force vanishes. The exact location of the point G is obtained as

$$(qs/L)(s/2) = qL/6 \qquad \Rightarrow \qquad s = L/\sqrt{3}$$

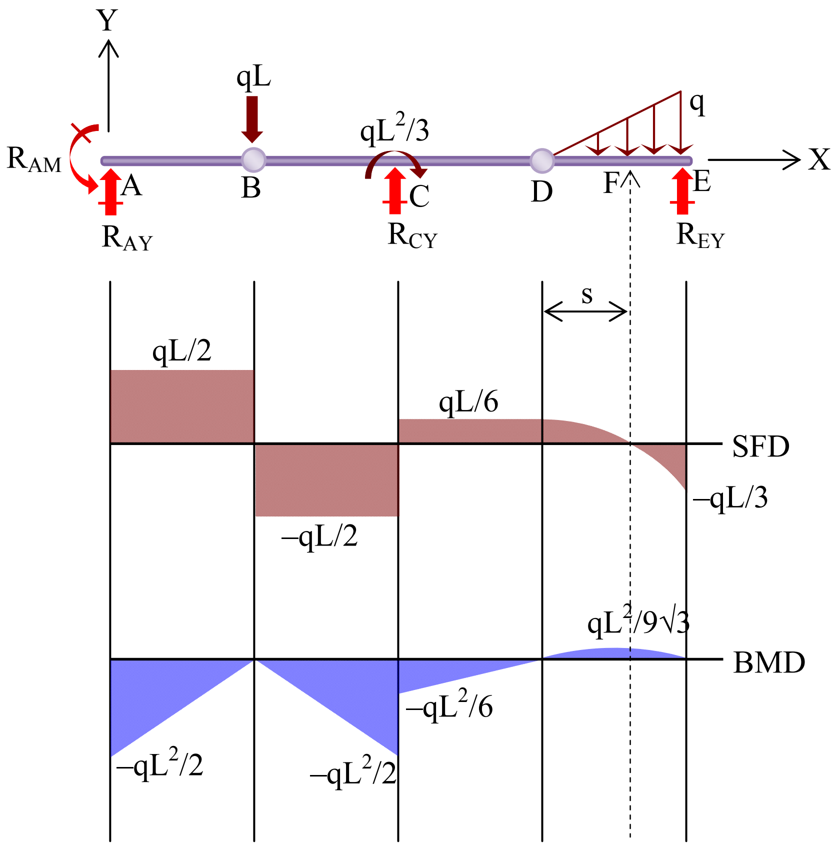
where s is the distance between the point D and point F. In addition, the value of the distributed at the point F is $q/\sqrt{3}$. The segment DE can now be separated into two sub-segments DF and FE; the shear force is positive for the first sub-segment while it is negative for the second sub-segment.



In the construction of the BMD, the differential formula (2.16), the integral formula (2.18), and the discontinuity conditions (2.20), (2.22) and (2.24) are utilized for each segment of the beam as described below:

- Segment AB
  - $M_A = -R_{AM} = -qL^2/2$
  - Area of the SFD over the segment is (qL/2)(L) + equation (2.18) $\Rightarrow M_{BL} = M_A + (qL/2)(L) = 0$
  - The shear force is constant and positive over a segment + equation (2.16) $\Rightarrow$ BMD over the segment is a rising straight line
- Segment BC
  - The negative concentrated force –qL is applied at point B + equation (2.20) $\Rightarrow$ there is no discontinuity of the bending moment at point B $\Rightarrow M_{BR} = M_{BL} = 0$
  - Area of the SFD over the segment is (–qL/2)(L) + equation (2.18) $\Rightarrow M_{CL} = M_{BR} + (-qL/2)(L) = -qL^2/2$
  - The shear force is constant and negative over a segment + equation (2.16) $\Rightarrow$ BMD over the segment is a dropping straight line
- Segment CD
  - The negative concentrated moment $-qL^2/3$ is applied at point C + equation (2.22) $\Rightarrow$ there is a jump of the bending moment at point C $\Rightarrow M_{CR} = M_{CL} - (-qL^2/3) = -qL^2/6$
  - Area of the SFD over the segment is (qL/6)(L) + equation (2.18) $\Rightarrow M_{DL} = M_{CR} + (qL/6)(L) = 0$
  - The shear force is constant and positive over the segment + equation (2.16) $\Rightarrow$ BMD over the segment is a rising straight line
- Segment DF
  - The distributed load changes its distribution at point D but it is still continuous $\Rightarrow$ there is no discontinuity of the bending moment at point D $\Rightarrow M_{DR} = M_{DL} = 0$
  - Area of the SFD over the segment is $2(qL/6)(L/\sqrt{3})/3$ + equation (2.18) $\Rightarrow M_{FL} = M_{DR} + qL^2/9\sqrt{3} = qL^2/9\sqrt{3}$
  - The shear force is positive and decreases monotonically in magnitude over a segment + equation (2.16) $\Rightarrow$ BMD over the segment is a rising and concave downward curve with a zero slope at point F

- Segment FE
  - There is no loading discontinuity at point F ⇒ there is no discontinuity of the bending moment at point F ⇒ $M_{FR} = M_{FL} = qL^2/9\sqrt{3}$
  - Area of the SFD over the segment is $-qL^2/9\sqrt{3}$ + equation (2.18) ⇒ $M_{EL} = M_{FR} - qL^2/9\sqrt{3} = 0$ ⇒ consistent with the condition at the right end of the beam
  - The shear force is negative and increases monotonically in magnitude over a segment + equation (2.16) ⇒ BMD over the segment is a dropping and concave downward curve



From above SFD and BMD, the maximum positive shear force is equal to qL/2 occurring at the entire segment AB; the maximum negative shear force is equal to –qL/2 occurring at the entire segment BC; the maximum positive bending moment is equal to $qL^2/9\sqrt{3}$ occurring at point F; and the maximum negative bending moment is equal to $qL^2/2$ occurring at points A and C.

2.5.6 Qualitative elastic curve

In this section, we provide basic idea for sketching qualitative elastic curve of a beam (a curve represented the deformed configuration of the neutral axis of the beam) under applied loads once the bending moment diagram (BMD) is determined. The key assumptions employed are those associated with Euler-Bernoulli beam theory; i.e. (i) beam is made from a linearly elastic material; (ii) plane section remains plane before and after undergoing deformation; (iii) no shear deformation; and (iv) no internal axial force and no axial deformation of the neutral axis. Schematics of Euler-Bernoulli beam before and after undergoing deformation are indicated in Figure 2.35. According to the kinematics assumption of deformation of the cross section, it is standard to represent the entire beam by its neutral axis. The elastic or deformed curve is therefore the deformed configuration of the neutral axis.

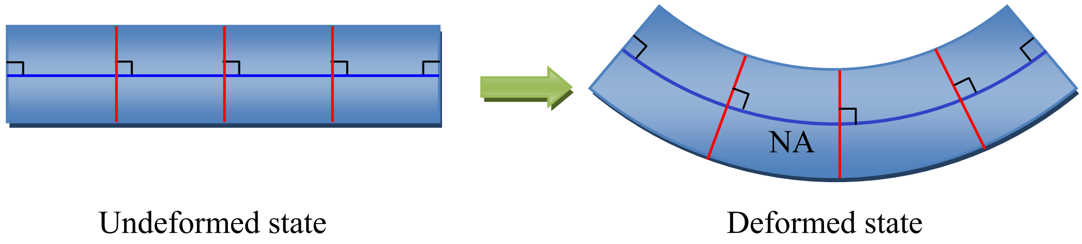


Figure 2.35: Schematics of undeformed and deformed configurations of a beam under bending

Let v(x) and θ(x) be the deflection and the rotation at any point x as shown schematically in Figure 2.36. The deflection v(x) is considered to be positive if it directs upward in the Y-direction and the rotation θ(x) is considered to be positive if it directs in a counter clockwise direction or in the Z-direction. By assuming that the deflection and the rotation are infinitesimally small in comparison with the length of the beam, the deflection v(x), the rotation θ(x), and the curvature κ(x) are related through the following relations

$$\theta(x) = \frac{dv}{dx} \tag{2.25}$$

$$\kappa(x) = \frac{d\theta}{dx} = \frac{d^2v}{dx^2} \tag{2.26}$$

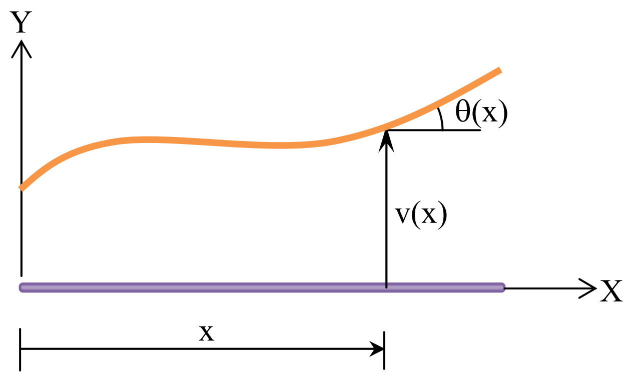


Figure 2.36: Schematics indicating the deflection and rotation at any point x

It is evident from the relation (2.26) that the curvature κ(x) is positive if the deflection is concave upward (i.e. $d^2v/dx^2 > 0$), negative if the deflection is concave downward (i.e. $d^2v/dx^2 < 0$), and zero if it is an inflection point (i.e. $d^2v/dx^2 = 0$). In addition, a direct consequence of the infinitesimal displacement and rotation assumption leads to zero displacement in the longitudinal direction of the beam or, equivalently, the preservation of the projected length of the beam onto its undeformed axis. This behavior implies that any point of the beam displaces only in a vertical direction or, more precisely, in a direction perpendicular to the axis of the beam.

By exploiting the kinematics assumption of the beam cross section, utilizing material constitutive, and computing the moment resultant of the cross section, it leads to a well-known moment-curvature relationship:

$$\kappa(x) = \frac{M(x)}{EI} \tag{2.27}$$

where E is the Young's modulus and I is the moment of inertia of the cross section. It can be deduced from the relations (2.26) and (2.27) that

- A segment of a beam possessing the *positive* bending moment undergoes a *positive* curvature and, as a result, leading to a concave *upward* elastic curve (see Figure 2.37);

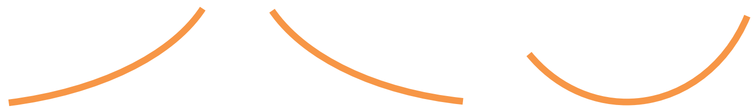


Figure 2.37: Schematics of concave upward elastic curve

- A segment of a beam possessing the *negative* bending moment undergoes a *negative* curvature and, as a result, leading to a concave *downward* elastic curve (see Figure 2.38);

Figure 2.38: Schematics of concave downward elastic curve

- A segment of a beam possessing the *zero* bending moment undergoes a *zero* curvature and, as a result, leading to a straight-line elastic curve; and
- A point within the beam where the bending moment changes sign at that point is an inflection point on the elastic curve.

To sketch the qualitative elastic curve, the following procedures are suggested:

- Construct BMD for the entire beam
- Use equation (2.27) to identify the shape of elastic curve at any segment of the beam
- Patch all segments of elastic curve together
- Check compatibility with all supports and internal releases.

It is important to emphasize that the deflection of the beam is continuous everywhere except at the shear releases and the rotation of the beam is continuous everywhere except at the moment releases or hinges. Figure 2.39 shown below indicate the discontinuity occurs at the shear release and the moment release.

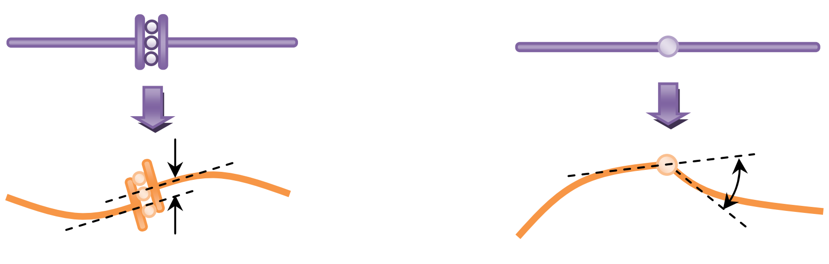
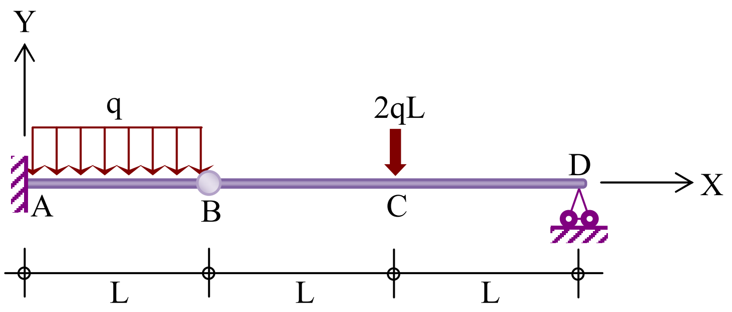
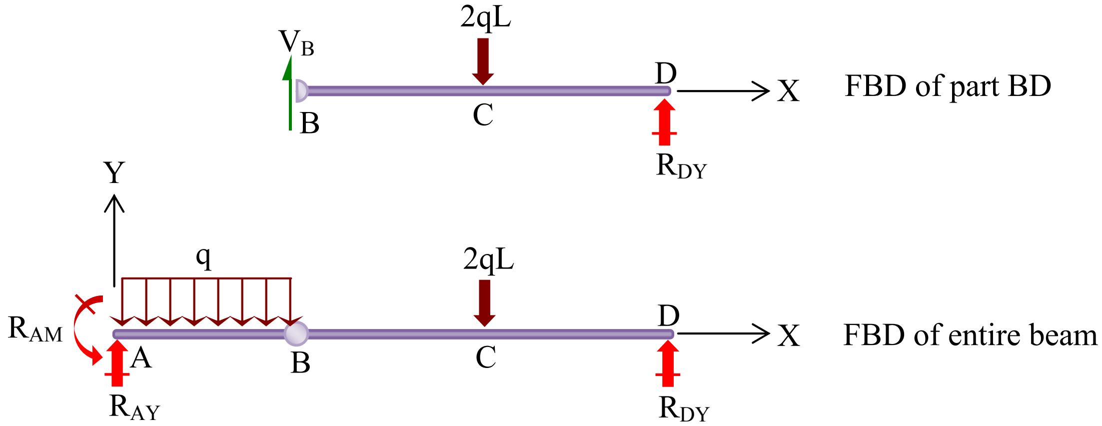


Figure 2.39: Schematics of deformed shape in the neighborhood of the shear and moment releases

Example 2.12 Compute all support reactions, sketch the SFD and BMD, and then sketch the elastic curve of a beam shown below



Solution The given beam is statically determinate (i.e. $r_a = 2 + 1 = 3$, $n_m = 1(2) = 2$, $n_j = 2(2) = 6$, $n_c = 1$, then DI = 3 + 2 – 4 – 1 = 0); thus, all support reactions and the internal forces at any location can be determined from static equilibrium. Since the number of support reactions is equal to 3, they cannot be obtained from equilibrium of the entire structure alone. By making a cut at the hinge B and then considering equilibrium of the right part (part BD), one of the reactions can be determined. The rest of the reactions can be computed from equilibrium of the entire beam. Details of calculation are shown below:



Equilibrium of a part BD

$[\Sigma M_{BR} = 0]$ ↺+ : $2R_{DY}L - (2qL)(L) = 0$

$R_{DY} = qL$ Upward

Equilibrium of entire beam

$[\Sigma M_A = 0]$ ↺+ : $R_{AM} + 3R_{DY}L - (q)(L)(L/2) - (2qL)(2L) = 0$

$R_{AM} = 3qL^2/2$ CCW

$[\Sigma F_Y = 0]$ ↑+ : $R_{AY} + R_{DY} - (q)(L) - 2qL = 0$

$R_{AY} = 2qL$ Upward

From the FBD of the entire structure, there are two points of loading discontinuity within the beam (excluding the two ends of the beam), e.g. points B and C. Specifically, the point B is a point where the distributed load is discontinuous and the point C is a point where the concentrated forced is applied. Therefore, we can divide the entire beam into three segments: AB, BC, and CD. Once the support reactions are determined, the shear force and the bending moment at the left end of the beam (i.e. point A) are known, i.e. $V_A = R_{AY} = 2qL$ and $M_A = -R_{AM} = -3qL^2/2$.

First, let us construct the SFD. The differential formula (2.15), the integral formula (2.17), and the discontinuity conditions (2.19), (2.21) and (2.23) are utilized for each segment as shown below:

- Segment AB
  - $V_A = 2qL$
  - A negative uniform distributed load $-q$ is applied over the segment + equation (2.17) $\Rightarrow V_{BL} = V_A + (-q)(L) = qL$
  - A negative uniform distributed load $-q$ is applied over the segment + equation (2.15) $\Rightarrow$ SFD over the segment is a dropping straight line
- Segment BC
  - The distributed load is discontinuous at point B + equation (2.23) $\Rightarrow$ there is no discontinuity of the shear force at point B $\Rightarrow V_{BR} = V_{BL} = qL$
  - There is no distributed load over the segment + equation (2.17) $\Rightarrow V_{CL} = V_{BR} + 0 = qL$
  - There is no distributed load over the segment + equation (2.15) $\Rightarrow$ SFD over the segment is a horizontal straight line
- Segment CD
  - The negative concentrated force $-2qL$ is applied at point C + equation (2.19) $\Rightarrow$ there is a jump of the shear force at point C $\Rightarrow V_{CR} = V_{CL} + (-2qL) = -qL$
  - There is no distributed load over the segment + equation (2.17) $\Rightarrow V_{DL} = V_{DR} + 0 = -qL \Rightarrow$ consistent with condition at the right end of the beam
  - There is no distributed load over the segment + equation (2.15) $\Rightarrow$ SFD over the segment is a horizontal straight line

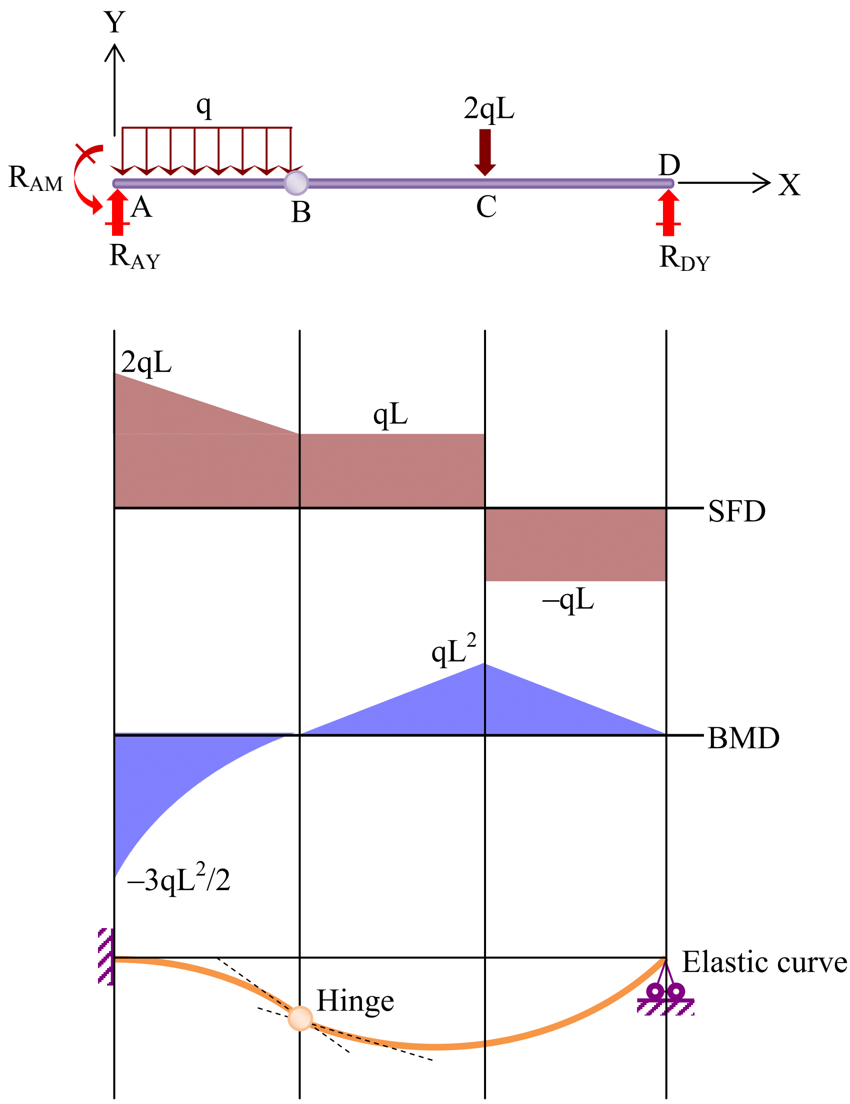
Once the SFD is obtained, the BMD can then be constructed. The differential formula (2.16), the integral formula (2.18), and the discontinuity conditions (2.20), (2.22) and (2.24) are utilized for each segment as described below:

- Segment AB
  - $M_A = -3qL^2/2$
  - Area of the SFD over the segment is $(2qL + qL)(L)/2$ + equation (2.18) $\Rightarrow M_{BL} = M_A + 3qL^2/2 = 0$
  - The shear force is positive and decreases monotonically in magnitude over the segment + equation (2.16) $\Rightarrow$ BMD over the segment is a rising and concave downward curve
- Segment BC
  - The distributed load is discontinuous at point B + equation (2.24) $\Rightarrow$ there is no discontinuity of the bending moment at point B $\Rightarrow M_{BR} = M_{BL} = 0$
  - Area of the SFD over the segment is $(qL)(L)$ + equation (2.18) $\Rightarrow M_{CL} = M_{BR} + (qL)(L) = qL^2$
  - The shear force is constant and positive over the segment + equation (2.16) $\Rightarrow$ BMD over the segment is a rising straight line
- Segment CD

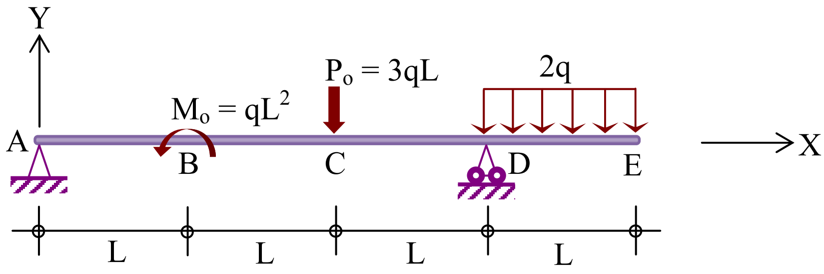
- The concentrated force is applied at point C + equation (2.20) ⇒ there is no discontinuity of the bending moment at point C ⇒ $M_{CR} = M_{CL} = qL^2$
- Area of the SFD over the segment is (–qL)(L) + equation (2.18) ⇒ $M_{DL} = M_{CR} + (-qL)(L) = 0$ ⇒ consistent with condition at the right end of the beam
- The shear force is constant and negative over the segment + equation (2.16) ⇒ BMD over the segment is a dropping straight line

From movement constraints provided by roller and fixed supports, a moment release, and the BMD shown below, we obtain the following information that is useful for sketching an elastic curve:

- Point A: fixed support ⇒ there is no rotation and deflection at this point
- Point B: hinge ⇒ the rotation is discontinuous at this point while displacement is still continuous
- Point D: roller support ⇒ there is no vertical displacement at this point while the rotation is allowed
- Segment AB: bending moment is negative ⇒ the elastic curve of this segment must be concave downward
- Segment BD: bending moment is positive ⇒ the elastic curve of this segment must be concave upward



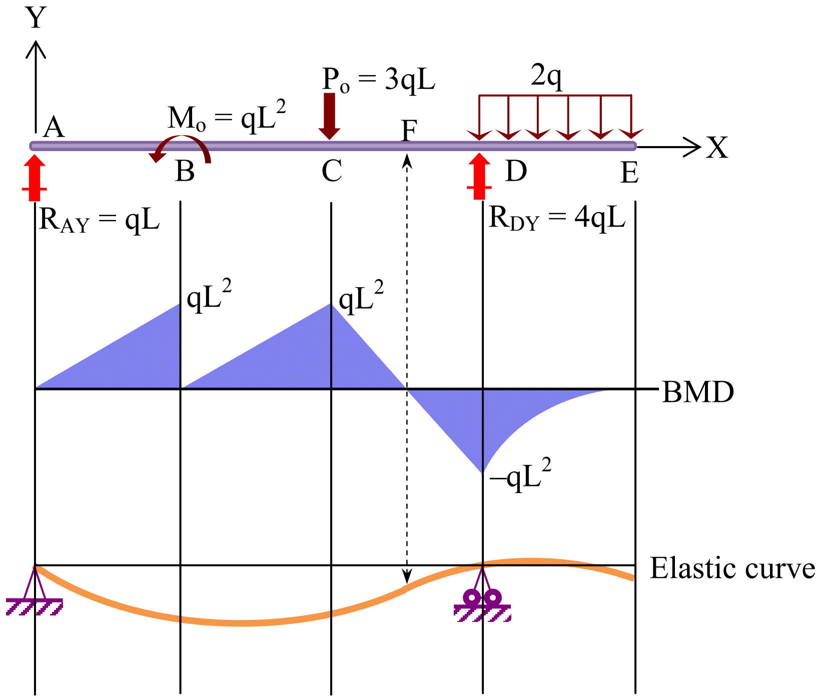
Example 2.13 Sketch the elastic curve of the beam shown in example 2.9



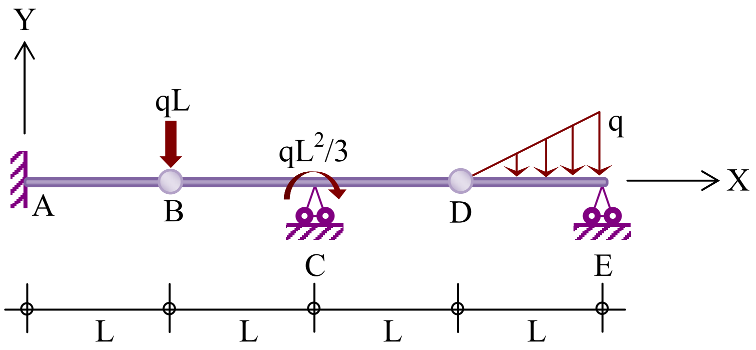
Solution From movement constraints provided by roller and pinned supports and the BMD obtained in example 2.9, we obtain the following information that is useful for sketching an elastic curve:

- Point A: pinned support ⇒ there is no vertical displacement at this point while the rotation is allowed
- Point D: roller support ⇒ there is no vertical displacement at this point while the rotation is allowed
- Point F: change sign of bending moment ⇒ inflection point
- Segment AB: bending moment is positive ⇒ the elastic curve of this segment must be concave upward
- Segment BC: bending moment is positive ⇒ the elastic curve of this segment must be concave upward
- Segment CF: bending moment is positive ⇒ the elastic curve of this segment must be concave upward
- Segment FD: bending moment is negative ⇒ the elastic curve of this segment must be concave downward
- Segment DE: Bending moment is negative ⇒ the elastic curve of this segment must be concave downward

The resulting elastic curve is shown below.



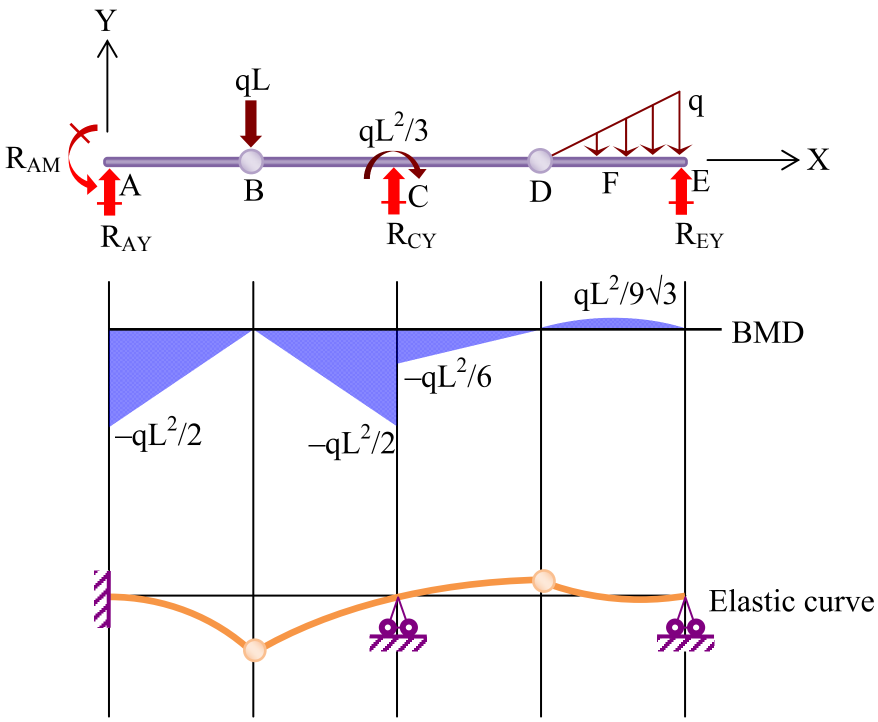
Example 2.14 Sketch the elastic curve of the beam shown in example 2.11



Solution From movement constraints provided by roller and fixed supports, moment releases, and the BMD obtained in example 2.11, we obtain the following information that is useful for sketching an elastic curve:

- Point A: fixed support ⇒ there is no rotation and deflection at this point
- Point C: roller support ⇒ there is no vertical displacement at this point while the rotation is allowed
- Point E: roller support ⇒ there is no vertical displacement at this point while the rotation is allowed
- Point B: hinge ⇒ the rotation is discontinuous at this point while displacement is still continuous
- Point D: hinge ⇒ the rotation is discontinuous at this point while displacement is still continuous
- Segments AB, BC, CD: bending moment is negative ⇒ the elastic curve of these segments must be concave downward
- Segment DE: bending moment is positive ⇒ the elastic curve of this segment must be concave upward

The resulting elastic curve is shown below.



2.6 Static Analysis of Frames

The primary objective of this section is to generalize the concept and techniques presented in the previous section to analyze a more general class of structures called *frames*. Basic quantities of interest from the static analysis are still the support reactions and the internal forces at any location. However, as will be clear in later discussion, the internal force of a frame is relatively more complex than that of a truss and a beam since it consists of all three components known as the axial force, the shear force and the bending moment. This section begins with a brief introduction on characteristics of frames and standard notations and sign convention commonly used. A brief overview on how to determine support reactions of statically determinate frames is addressed along with some useful remarks. In the analysis for the internal forces, both the method of sections and the method of differential and integral formula are outlined. While these two methods are similar in accord to those employed in the analysis of beams, they possess an additional feature capable of treating structures whose internal force fully containing the axial force, the shear force and the bending moment. Finally, some guidelines for sketching a qualitative elastic curve for frames are summarized. At the end of this section, some examples are also presented to clearly demonstrate all techniques outlined.

2.6.1 Characteristics of frames

An idealized structure is called a *planar frame* if and only if (i) all members form a two-dimensional structure, (ii) members are connected by rigid or frame joints (full or partial moment releases are allowed for certain joints and this can be viewed as rigid joints supplying by moment releases), and (iii) applied loads form a system of general two-dimensional forces and moments (i.e. it includes both transverse and longitudinal loads). Examples of planar frames following the above definition are shown in Figure 2.40. Note that there is no restriction on the type of supports present in frames, i.e. roller supports, pinned supports, guide supports and fixed supports are allowed. The pinned support and fixed support in frames contain two and three components of the reaction, respectively; the restriction on the number of reactions as in the case of beams is now removed. Note in addition that a one-dimensional structure shown in Figure 2.40 is also classified as a frame since it is subjected to both transverse and longitudinal loads.

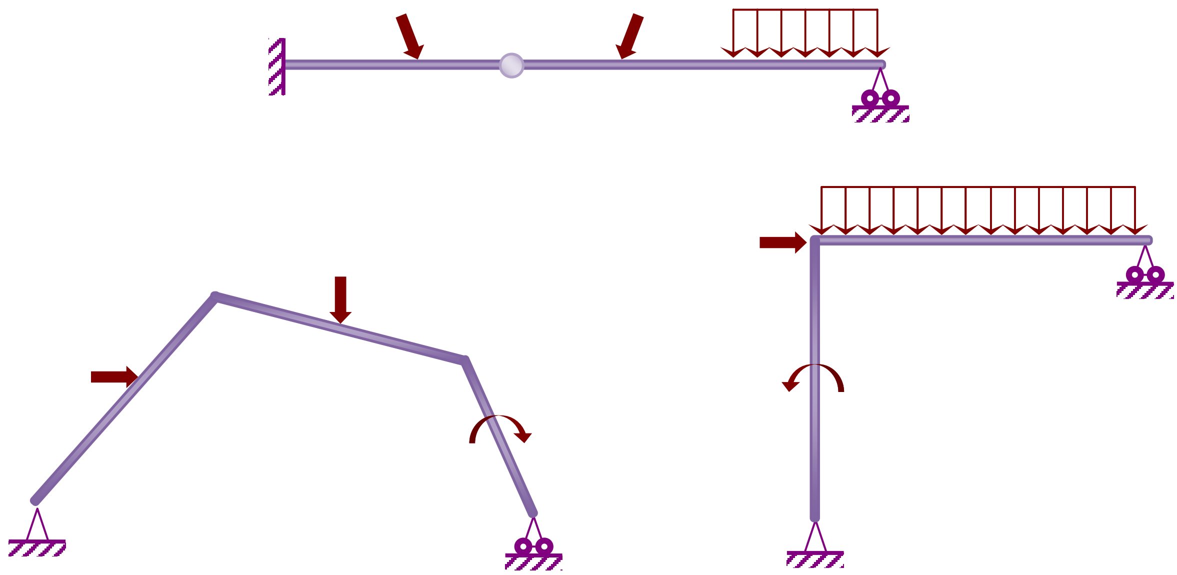


Figure 2.40: Schematic of some statically determinate frames

As a consequence of the above definition, the internal forces at any location of the beam can be represented by three components of resultants called the *axial force*, the *shear force* and the *bending moment*. The last two components of the internal force are defined in the same fashion as those for the beams and the first component, i.e. the axial force, is the resultant force in the direction parallel to the axis of the member, see Figure 2.41 for clarity. Like a beam member, the axial force, the shear force and the bending moment along a member of the frame are in general not constant but vary as a function of position. Note also that the displacement and rotation at any point within the frame that contain no internal release are always continuous; for instance, the angle between any two members connected by a rigid joint is preserved and there is no gap and overlapping at any point containing no internal release after undergoing deformation.

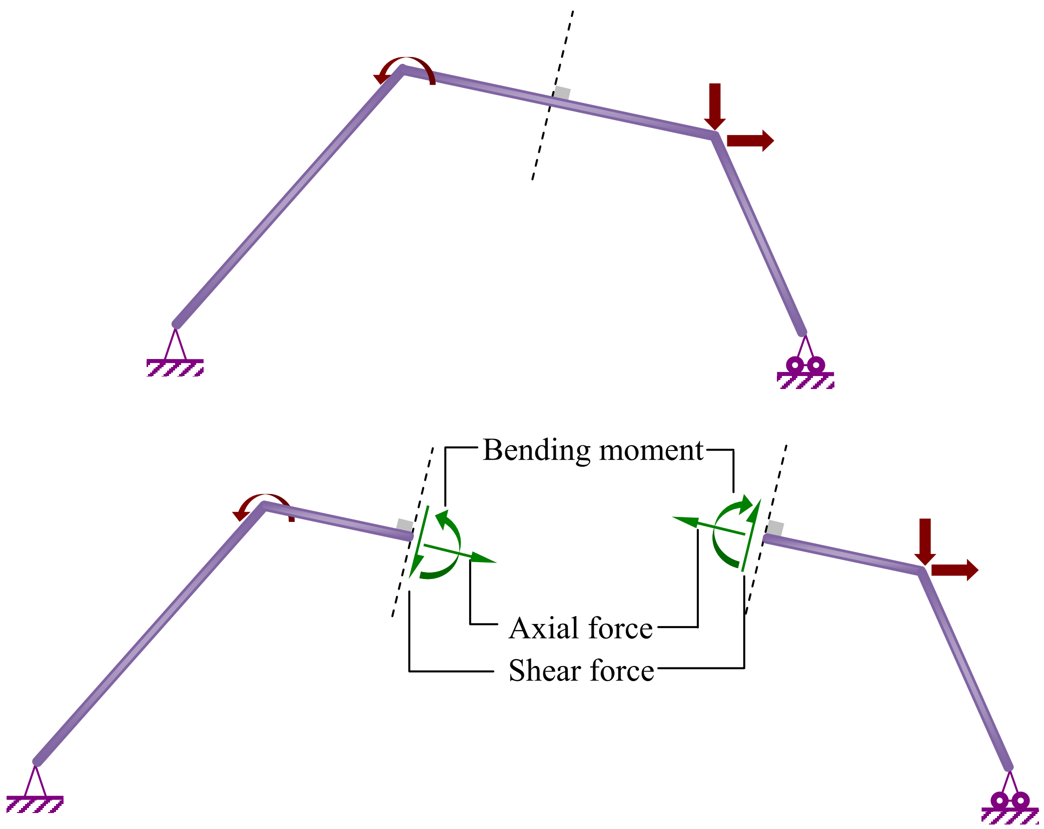


Figure 2.41: Schematic indicating three components of the internal force in planar frame

2.6.2 Sign and convention

Since members of a frame can possess different orientations, it is generally impossible to find a single reference Cartesian coordinate system with one of its axes directing along the axis of all members as in the case of beams. As a result, it is common to employ two different types of reference coordinate systems, one termed a *global coordinate system* and the other termed a *local coordinate system*. The global coordinate system is a single coordinate system used as a reference of the entire structure. A choice of the global coordinate system is not unique; in particular, an orientation of the reference axes and a location of its origin can be chosen arbitrarily or for convenience. The global reference axes are labeled by X, Y, and Z with their directions following the right-hand rule. For a two-dimensional structure, the global coordinate system is typically oriented such that the Z-axis directs normal to the plane of the structure. The local coordinate system is a coordinate system defined for an individual member. The local reference axes are labeled by x, y, and z. The local coordinate system for each member is commonly defined based on the geometry and orientation of that member. Specifically, the origin is taken at one end of the member; the x-axis directs along the axis of the member; the z-axis directs normal to the plane of the structure (align with the Z-axis), and the y-axis follows from the right-hand rule. An example of global and local coordinate systems of a planar frame is shown in Figure 2.42.

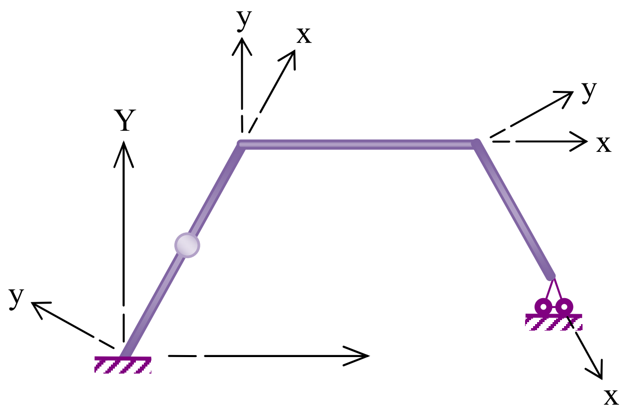


Figure 2.42: Global and local coordinate systems of a planar frame.

The sign convention and notations for support reactions of frames are defined in a similar fashion as those for trusses and beams with reference to the global coordinate system. For instance, reactions at the fixed support of a frame shown in Figure 2.43 are denoted by R_{AX}, R_{AY} and R_{AM} where the first two symbols stand for a force reaction in the X-direction and a force reaction in the Y-direction and the last symbol stands for a moment reaction in the Z-direction, and a force reaction in the Y-direction of the roller support located at a point B is denoted by R_{BY}. In the analysis for support reactions, it is common to assume positive support reactions in a sketch of the FBD. Once results are obtained, the actual direction of each reaction can be decided from their sign; specifically, if the computed reaction is positive, the assumed direction is correct but, if the computed reaction is negative, the actual direction is opposite to the assumed direction.

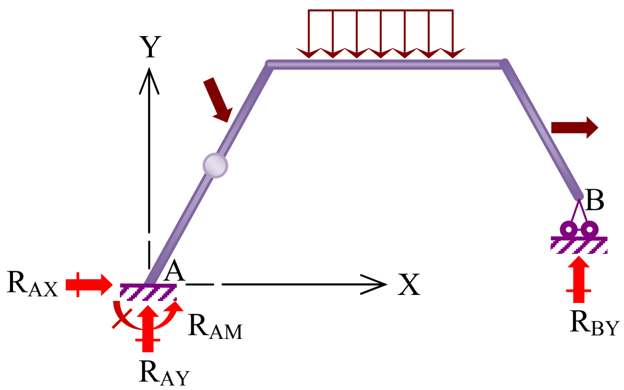


Figure 2.43: Schematic showing sign convention and notations of support reactions of a frame

The sign convention for the shear force and the bending moment for a frame member depend primarily on a choice of the local coordinate system. Once the local coordinate system is selected, the sign convention is defined in the same way as that for a beam (the local coordinate system x-y of a frame member can be viewed as the X-Y axis of a beam member):

- The shear force at a particular point A is denoted by a symbol V_A and the shear force as a function of position x along the member is denoted by V(x). The shear force at any point is considered to be positive if and only if it tends to rotate an infinitesimal element in the neighborhood of that point in the negative z-direction otherwise it is negative. The positive and negative shear forces are shown in Figure 2.44.

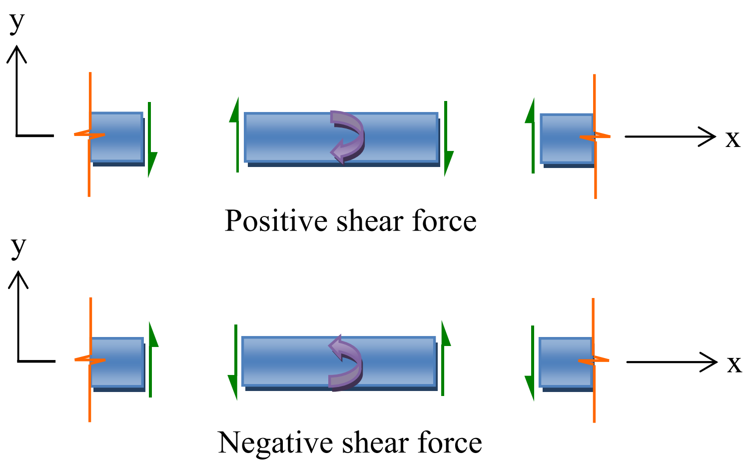


Figure 2.44: Schematic indicating positive and negative sign convention for shear force

- The bending moment at a particular point A is denoted by a symbol M_A and the bending moment as a function of position x along the member is denoted by M(x). The bending moment at any point is considered to be positive if and only if it produces a compressive stress at the top and produces a tensile stress at the bottom; otherwise it is negative. Note that the top and bottom sides of the member are defined based on the local coordinate system as shown in Figure 2.45.

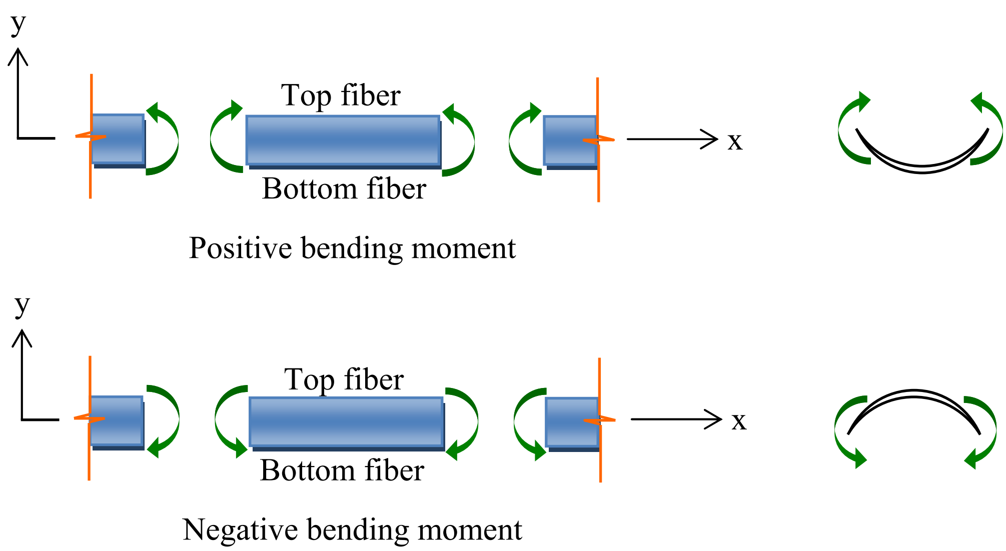


Figure 2.45: Schematic indicating positive and negative sign convention for bending moment

Similar to the axial force in a truss member, the sign convention for the axial force in a frame member is defined based primarily upon the characteristic of the axial deformation, thus rendering it independent of the local coordinate system (or the member orientation). In particular, the axial force at a particular point A is denoted by a symbol F_A and the axial force as a function of position x along the member is denoted by F(x). The axial force at any point is considered to be positive if and only if it produces an elongation in the neighborhood of that point or it is in tension otherwise it is negative. The positive axial force and the negative axial force are shown schematically in Figure 2.46.

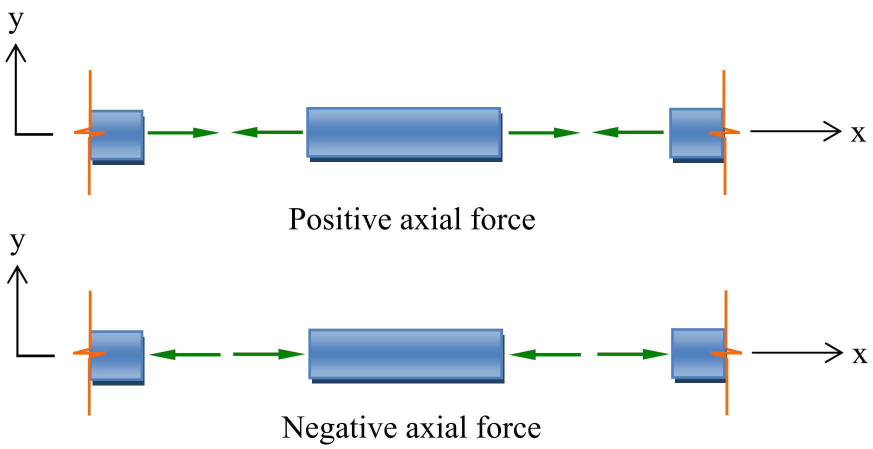


Figure 2.46: Schematic indicating positive and negative sign convention for axial force

2.6.3 Determination of support reactions

Determination of support reactions of statically determinate frame follows the same procedures described in the section 2.3. For frames containing only three components of support reactions, such unknown reactions can readily be determined from equilibrium of the entire structure. For example, support reactions {R_{AX}, R_{AY}, R_{BY}} of a frame shown in Figure 2.47 can be obtained as follows:

- the reaction R_{AX} is obtained from equilibrium of forces in Y-direction;
- the reaction R_{BY} is obtained from equilibrium of moments about a point A; and
- the reaction R_{AY} is obtained from equilibrium of forces in Y-direction.

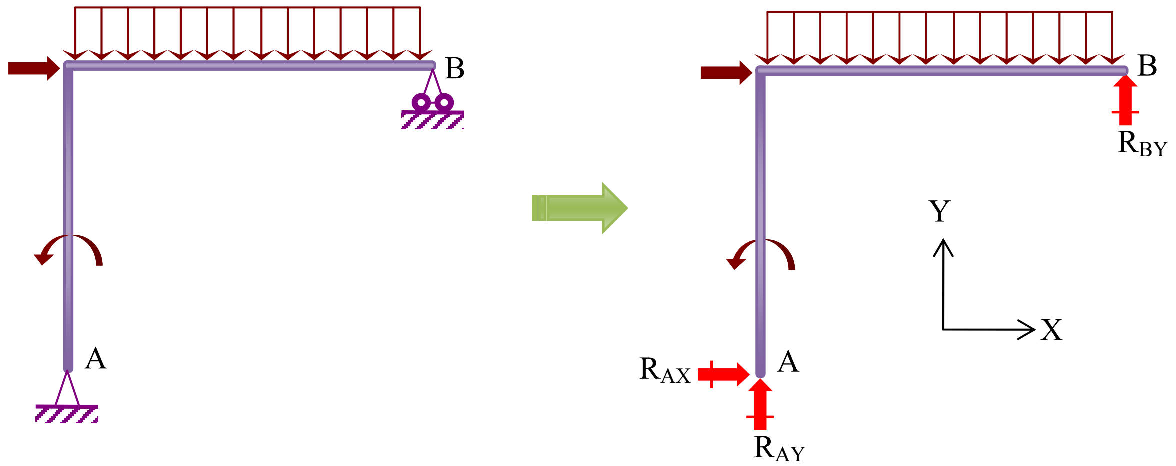


Figure 2.47: Schematic of a planar frame and its FBD

For various statically determinate frames, more than three components of support reactions may be present; for instance, both the frame shown in Figure 2.43 and a frame shown in Figure 2.48 contain four and five components of support reactions, respectively. For this particular case, the consideration of equilibrium of the entire structure provides only three independent equations and this is insufficient to determine all unknown reactions. Since the structure is statically determinate, there must be some internal releases that supply additional conditions, when combined with existing equilibrium equations, adequate for resolving all unknowns. Such extra or additional conditions available at the internal releases are commonly be set up in a form of equilibrium equations of parts of the structure resulting from introducing proper fictitious cuts.

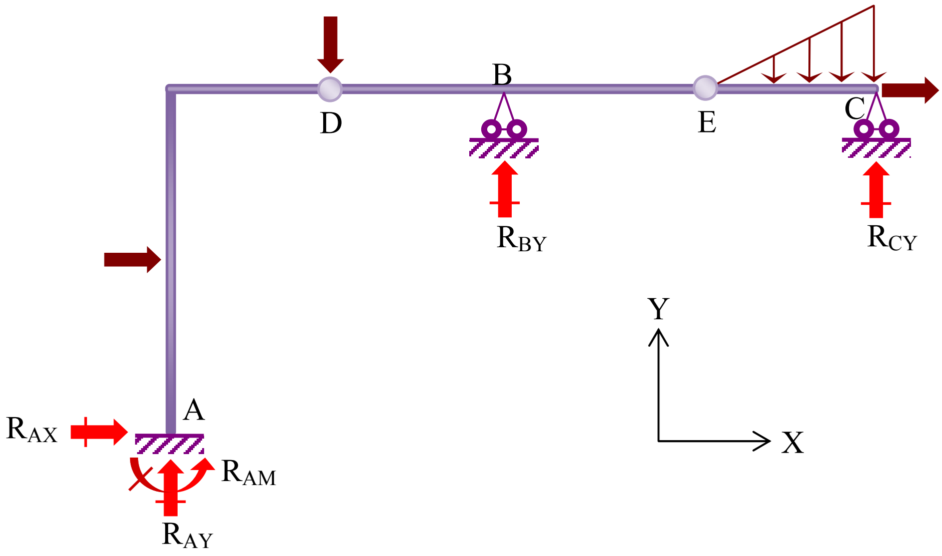


Figure 2.48: Schematic of a statically determinate frame containing five support reactions

To clearly demonstrate the procedures, let us consider a frame shown in Figure 2.48. This structure is obviously statically determinate (i.e., $r_a = 5$, $n_m = 3(3) = 9$, $n_j = 4(3) = 12$, $n_c = 2 \rightarrow DI = 5 + 9 - 12 - 2 = 0$) and this therefore ensures that all five support reactions can be obtained only from static equilibrium. To solve for all reactions $\{R_{AM}, R_{AX}, R_{AY}, R_{BY}, R_{CY}\}$, two strategies may be used. The first strategy employs additional equilibrium equations of parts of the structure along with equilibrium of the entire structure. Specifically, we first introduce a cut at a hinge E and consider equilibrium of the right part of the structure (see FBD in Figure 2.49(a)); next, we introduce a cut at a point just to the right of a hinge D and consider equilibrium of the right part of the structure (see FBD in Figure 2.49(b)); and, finally, we consider equilibrium of the entire structure. Details of equilibrium equations employed are shown below:

- the reaction R_{CY} is obtained from equilibrium of moments about the point E of the right part of the frame shown in Figure 2.49(a);
- the reaction R_{BY} is obtained from equilibrium of moments about a point D_R of the right part of the frame shown in Figure 2.49(b);
- the reaction R_{AM} is obtained from equilibrium of moments about a point A of the entire frame shown in Figure 2.49(c);
- the reaction R_{AX} is obtained from equilibrium of forces in the X-direction of the entire structure; and
- the reaction R_{AY} is obtained from equilibrium of forces in the Y-direction of the entire structure.

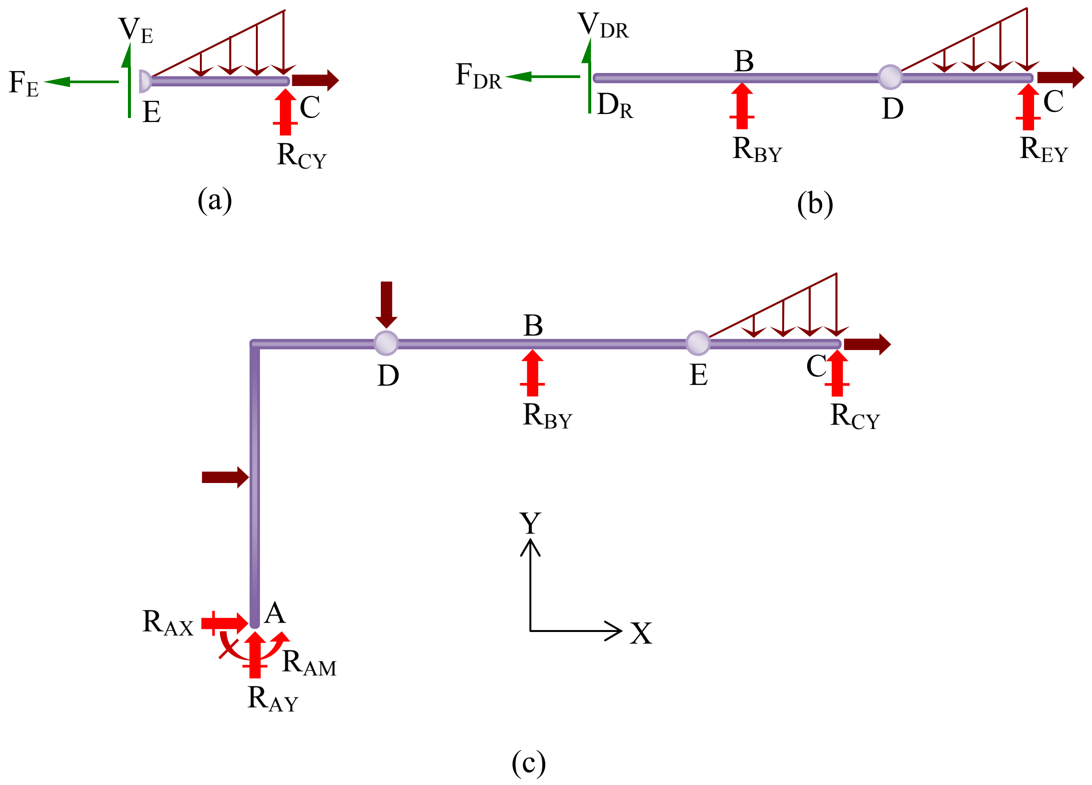


Figure 2.49: (a) FBD of the right part of the frame when a cut is made at hinge C, (b) FBD of the right part of the frame when a cut is made at the point just to the right of hinge D, and (c) FBD of the entire frame

The second strategy employs equilibrium equations set up for all parts of the structure. Specifically, we first introduce two cuts simultaneously, one at the hinge E and the other at point just to the right of the hinge D. With these two cuts, the structure is divided into three parts whose the FBDs are shown in Figure 2.50. While we introduce four extra unknowns $\{F_{DR}, V_{DR}, F_E, V_E\}$ at the cuts and the total number of unknowns becomes 5 + 4 = 9, it is equal to the number of equilibrium equations that can be set up for the three parts (3 + 3 + 3 = 9). To obtain all reactions without solving a system of nine linear equations, we can consider equilibrium of each part as follow:

- the reaction R_{CY} is obtained from equilibrium of moments about the point E of the right part of the frame shown in Figure 2.50(c), and
- the axial force F_E is obtained from equilibrium of forces in X-direction of the right part of the frame shown in Figure 2.50(c), and
- the shear force V_E is obtained from equilibrium of forces in Y-direction of the right part of the frame shown in Figure 2.50(c), and
- the reaction R_{BY} is obtained from equilibrium of moments about a point B_R of the middle part of the frame shown in Figure 2.50(b), and
- the axial force F_{DR} is obtained from equilibrium of forces in X-direction of the middle part of the frame shown in Figure 2.50(b), and
- the shear force V_{DR} is obtained from equilibrium of forces in Y-direction of the middle part of the frame shown in Figure 2.50(b), and
- the reaction R_{AM} is obtained from equilibrium of moments about a point A of the left part of the frame shown in Figure 2.50(a), and
- the reaction R_{AX} is obtained from equilibrium of forces in X-direction of the left part of the frame shown in Figure 2.50(a).

- the reaction R_{AY} is obtained from equilibrium of forces in Y-direction of the left part of the frame shown in Figure 2.50(a).

As is apparent, while both strategies yield identical results, one may prefer the first strategy since it is not required to solve for the intermediate unknowns $\{F_{DR}, V_{DR}, F_E, V_E\}$ introduced at the cuts.

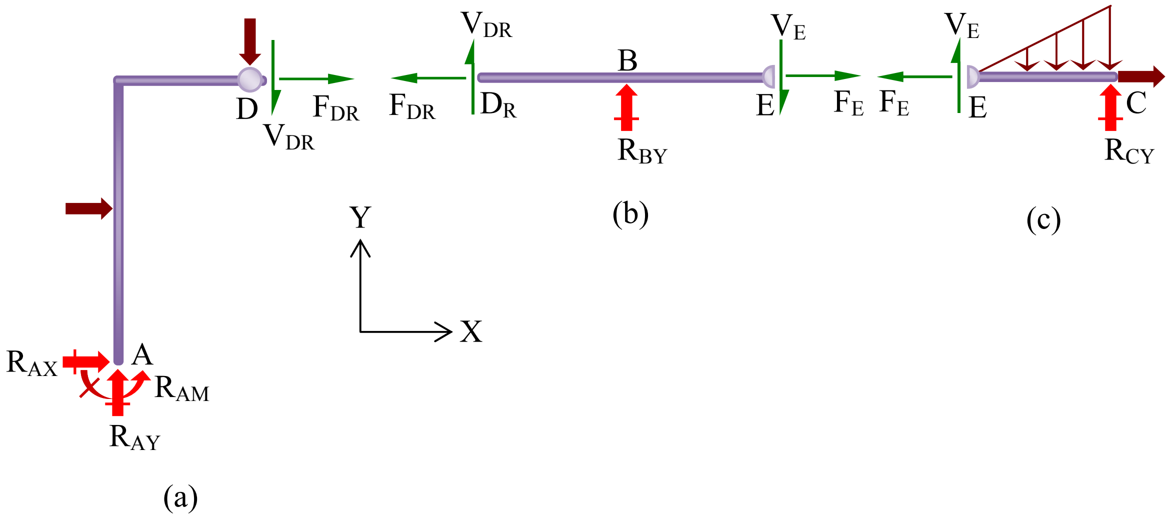


Figure 2.50: Free body diagrams of three parts of the frame resulting from two cuts at point just to the right of hinge D and at hinge E

2.6.4 Method of sections

To determine the internal forces (i.e. axial force, shear force, and bending moment) at a particular location of a statically determinate frame, the method of sections similar to that used in the case of beams can be employed. A fictitious cut is properly made to access the internal forces at a specific location of interest and equilibrium of parts of the structure resulting from that cut is then enforced to determine all unknown internal forces. Note in particular that three unknown internal forces (i.e. axial force, shear force, and bending moment) are introduced at each cut except at the internal releases where certain components of the internal force vanish, and that three independent equilibrium equations (e.g. $\Sigma F_X = 0$; $\Sigma F_Y = 0$; and $\Sigma M_{AZ} = 0$ or other equivalent sets) can be set up for each part resulting from the cut.

Procedures for obtaining the internal force at a particular location of a frame can be summarized below (see also a frame shown in Figure 2.51 to clarify such procedures):

- Determine all support reactions
- Introduce a fictitious cut at a point P (point where the internal force is to be determined) and then separate the frame into two parts
- Choose one of the two parts that seems to involve less computation
- Sketch the FBD of a selected part; the positive sign convention of the internal forces follows the local coordinate system of a member containing the point P
- Consider equilibrium of the selected part and this yields three independent equations. For instance, the axial force F_P can directly be obtained from equilibrium of forces in the X-direction; the shear force V_P can directly be obtained from equilibrium of forces in the Y-direction; and the bending moment M_P can be obtained from equilibrium of moment about the point P.

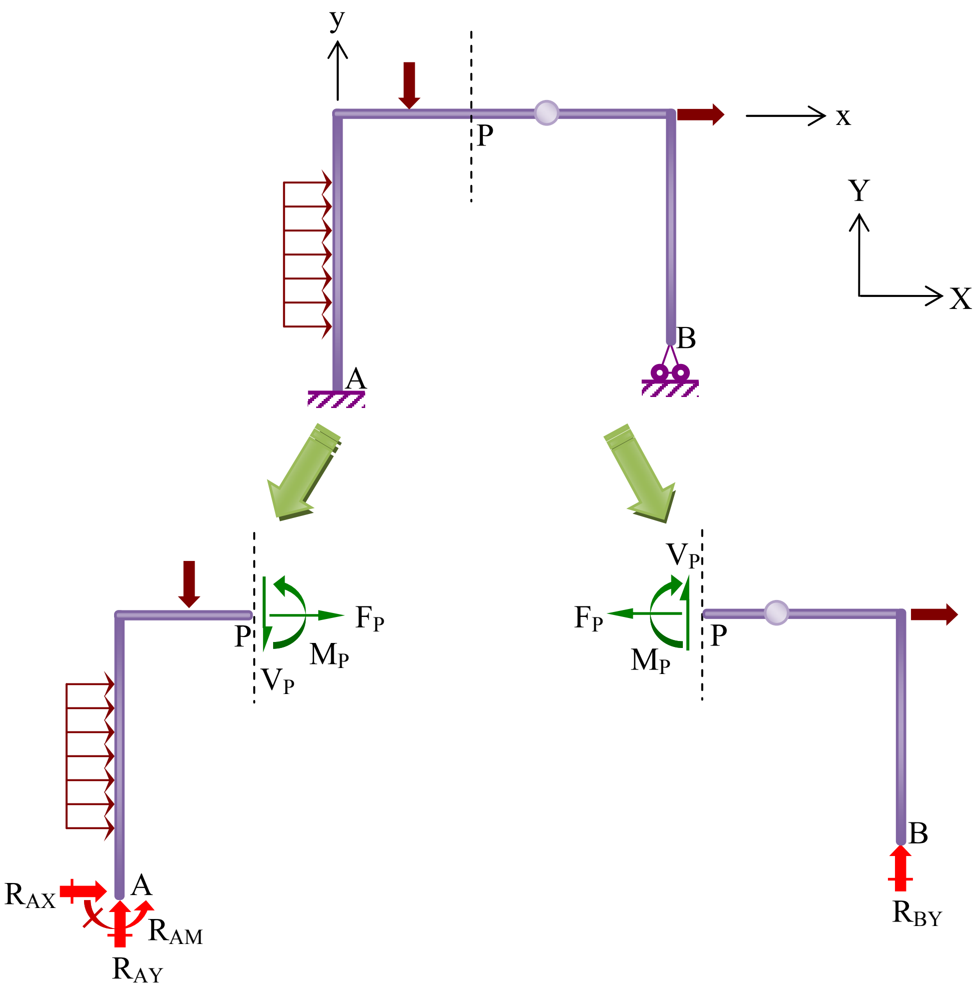


Figure 2.51: Schematic indicating a cut made to access the axial force, shear force and bending moment at point P and FBDs of the two parts resulting from the cut

Similar to the case of beams, the method of sections can also be used to determine the axial force, the shear force and the bending moment at any point x of each frame member, i.e. F(x), V(x) and M(x). Procedures to obtain F(x), V(x) and M(x) are similar to those described above except that a cut must be made at any point x instead of a specific location. Similar to the SFD and BMD, a graph of F(x) plotted along the local x-axis of each member is known as an *axial force diagram* (AFD). Note however that the construction of the AFD, SFD and BMD by the method of sections is somewhat cumbersome especially when there are many points of loading discontinuity and points where members change their direction.

2.6.5 Method of differential and integral formula

An alternative technique to the method of sections for the construction of the AFD, SFD and BMD of a frame is the method of differential and integral formula. This technique is based mainly upon two sets of equations, one associated with equilibrium equations in a differential form and the other corresponding to equilibrium equations in an integral form, and some special discontinuity conditions at points of loading discontinuity. Note that this technique is similar in accord to that presented in the section 2.5.5 for beams except that one equation associated with equilibrium of forces in the direction of the member axis is added due to the presence of the axial force.

2.6.5.1 Equilibrium equations in differential form

Consider a frame that is in equilibrium with applied loads as shown schematically in Figure 2.52(a). Let us focus on a particular member (with a local coordinate system x-y) and let introduce two cuts, one at the local coordinate x and the other at the local coordinate x + dx. An infinitesimal element dx is then separated from the structure and its FBD is shown in Figure 2.52(b). The axial force, shear force and bending moment at x are denoted by F(x), V(x) and M(x), respectively; the axial, shear force and bending moment at x + dx are denoted by F(x) + dF, V(x) + dV and M(x) + dM, respectively, where dF, dV and dM are increments of axial force, shear force and bending moment; and the distributed transverse and longitudinal loads are denoted by q_y and q_x, respectively. Note that the all components of the internal forces follow the sign convention defined in the section 2.6.2 while the distributed loads q_y and q_x are considered to be positive if their direction is along the y-axis and x-axis, respectively.

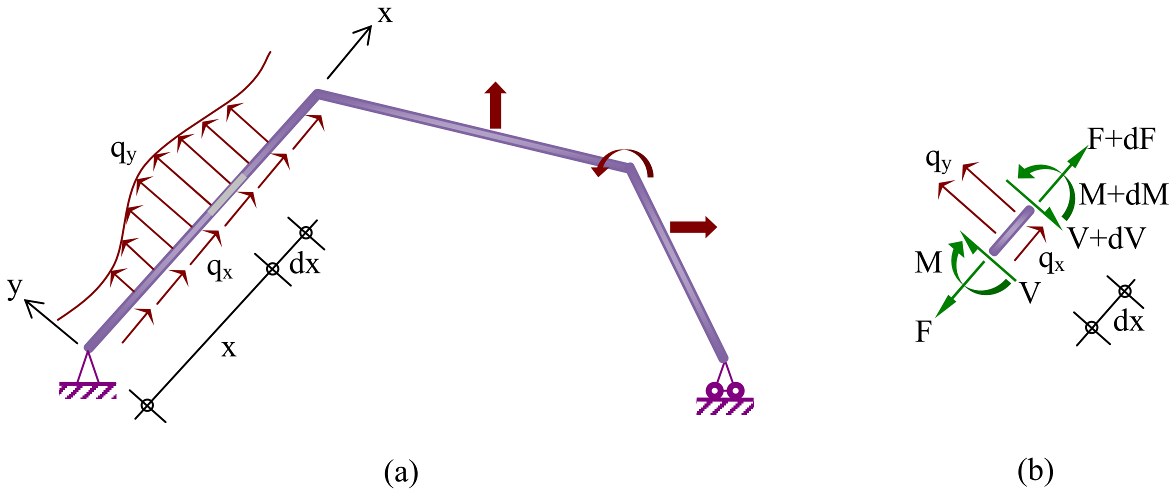


Figure 2.52: (a) Schematic of a frame subjected to applied loads and (b) FBD of an infinitesimal element dx

By enforcing static equilibrium of the infinitesimal element dx shown in Figure 2.52(b) and then taking appropriate limit process, we obtain the following three equilibrium equations in a differential form:

$$\Sigma F_x = 0 \quad \Rightarrow \quad \frac{dF(x)}{dx} = -q_x(x) \tag{2.28}$$

$$\Sigma F_y = 0 \quad \Rightarrow \quad \frac{dV(x)}{dx} = q_y(x) \tag{2.29}$$

$$\Sigma M_z = 0 \quad \Rightarrow \quad \frac{dM(x)}{dx} = V(x) \tag{2.30}$$

It is important to emphasize that the equation (2.28) is valid at any point x where the distributed longitudinal load q_x is continuous and it is free of a longitudinal concentrated force; the equation (2.29) is valid at any point x where the distributed transverse load q_y is continuous and it is free of a transverse concentrated force; and the equation (2.30) is valid at any point x such that the shear force is continuous and it is free of a concentrated moment.

The first equation (2.28) implies that the spatial rate of change of the axial force or the "slope of the axial force diagram" at any point is equal to the distributed longitudinal load q_x acting to that point. The second equation (2.29) implies that the spatial rate of change of the shear force or the "slope of the shear force diagram" at any point is equal to the distributed transverse load q_y acting that point. The last equation (2.30) implies that the spatial rate of change of the bending moment or the "slope of the bending moment diagram" at any point is equal to the shear force at that point. Similar to the case of beams, the three differential relations (2.28)-(2.30) are very useful for identifying the type of curve connecting any two points in the AFD, the SFD and the BMD (see the section 2.5.5.1 for more details on some specific types of curves).

2.6.5.2 Equilibrium equations in integral form

Now, let A and B be any two points within a frame member and let x_A and x_B are their x-coordinates with respect to the local coordinate system of the member. By directly integrating equation (2.28) from the point A to the point B, we obtain the integral formula

$$F_B = F_A - \int_{x_A}^{x_B} q_x \, dx = F_A - Q_{AB}^{long} \tag{2.31}$$

where F_A and F_B are the axial force at the point A and the point B, respectively, and Q_{AB}^{long} is the sum of distributed longitudinal load q_x over the segment AB. This equation implies that the axial force at the point B can be obtained by subtracting the total longitudinal load over the segment AB to the axial force at the point A. Note that the equation (2.31) is valid if the segment AB is free of no concentrated longitudinal force and that the sign convention of the total load Q_{AB}^{long} follows that of the distributed load q_x.

Similarly, by directly integrating equation (2.29) from the point A to the point B, we obtain the integral formula

$$V_B = V_A + \int_{x_A}^{x_B} q_y \, dx = V_A + Q_{AB}^{tran} \tag{2.32}$$

where V_A and V_B are the shear force at the point A and the point B, respectively and Q_{AB}^{tran} is the sum of all distributed transverse load q_y over the segment AB. This equation implies that the shear force at the point B can be obtained by adding the total transverse load over the segment AB to the shear force at the point A. Note that the equation (2.32) is valid if the segment AB is free of no concentrated transverse force and that the sign convention of the total load Q_{AB}^{tran} follows that of the distributed load q_y.

Finally, by directly integrating equation (2.30) from the point A to the point B, we obtain the integral formula

$$M_B = M_A + \int_{x_A}^{x_B} V \, dx = M_A + \text{AreaV}_{AB} \tag{2.33}$$

where M_A and M_B are the bending moment at the point A and the point B, respectively and AreaV_{AB} denotes the area of the shear force diagram over the segment AB. This equation implies that the bending moment at the point B can be obtained by adding the area of the shear force diagram over the portion AB to the bending moment at the point A. Note that the equation (2.33) is valid if the segment AB is free of no concentrated moment and that the sign convention of the area $AreaV_{AB}$ follows that of the shear force.

2.6.5.3 Discontinuity of axial force, shear force and bending moment

As already pointed out, the differential formula (2.28)-(2.30) and the integral formula (2.31)-(2.33) are not valid for points of loading discontinuity or segments that contain those points, e.g. points where the concentrated force is applied, points where the concentrated moment is applied, and points where the distributed load is discontinuous. Knowledge of discontinuity conditions at such points is required in the construction of the AFD, SFD, and BMD.

First, let us investigate the discontinuity condition at the location where a concentrated longitudinal force is applied. Let P_o be such a concentrated force applied to a point A of a particular frame member (it is emphasized again that this force is considered to be positive if it directs in the local x-direction of the member otherwise it is negative). By introducing two cuts at a point just to left and a point just to the right of the point A and then considering equilibrium of a resulting infinitesimal element (its free body diagram is shown in Figure 2.53 along with taking appropriate limit process, we obtain the discontinuity conditions of the axial force, shear force and bending moment:

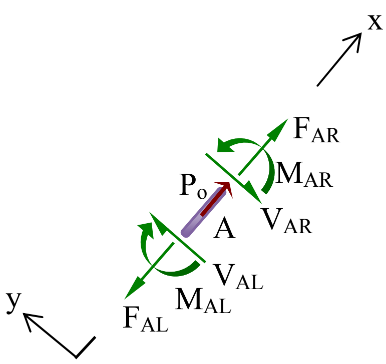


Figure 2.53: FBD of infinitesimal element containing a point where concentrated longitudinal force is applied

$$F_{AR} = F_{AL} - P_o \tag{2.34}$$

$$V_{AR} = V_{AL} \tag{2.35}$$

$$M_{AR} = M_{AL} \tag{2.36}$$

where $\{F_{AR}, V_{AR}, M_{AR}\}$ and $\{F_{AL}, V_{AL}, M_{AL}\}$ are the axial force, shear force and bending moment at a point just to the right and a point just to the left of the point A, respectively. It is evident from (2.34)-(2.36) that, at the location where a concentrated longitudinal force is applied, the axial force is discontinuous with the magnitude of the jump equal to the magnitude of the concentrated longitudinal force while the shear force and the bending moment are still continuous. In particular, the axial force experiences a positive jump if the concentrated longitudinal force is negative (or directs in the opposite local x-direction) and it experiences a negative jump if the concentrated longitudinal force is positive (or directs in the local x-direction).

Next, let us consider the discontinuity conditions at a location where a concentrated transverse force is applied. Let P_o be such a concentrated force applied to a point A of a particular frame member (it is emphasized again that this force is considered to be positive if it directs in the local y-direction of the member otherwise it is negative). By introducing two cuts at a point just to the left and a point just to the right of the point A and then considering equilibrium of a resulting infinitesimal element (its free body diagram is shown in Figure 2.54) along with taking appropriate limit process, we obtain the discontinuity conditions of the axial force, shear force and bending moment:

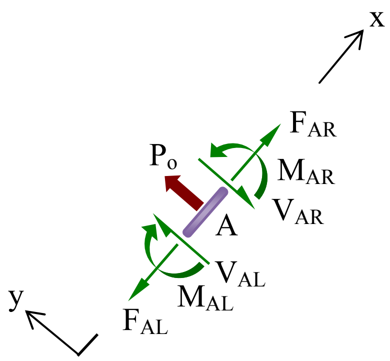


Figure 2.54: FBD of infinitesimal element containing a point where concentrated transverse force is applied

$$F_{AR} = F_{AL} \tag{2.37}$$

$$V_{AR} = V_{AL} + P_o \tag{2.38}$$

$$M_{AR} = M_{AL} \tag{2.39}$$

Equations (2.37)-(2.39) imply that, at the location where a concentrated transverse force is applied, the shear force is discontinuous with the magnitude of the jump equal to the magnitude of the concentrated transverse force while the axial force and the bending moment are still continuous. Unlike the previous case, the shear force experiences a positive jump if the concentrated transverse force is positive (or directs in the local y-direction) and it experiences a negative jump if the concentrated transverse force is negative (or directs in the opposite local y-direction).

Next, let us consider the discontinuity conditions at a location where a concentrated moment M_o is applied. Let M_o be a concentrated moment applied to a point A (this moment is considered to be positive if it directs in a counter clockwise direction or in the local z-direction of the member otherwise it is negative). By introducing two cuts at a point just to the left and a point just to the right of the point A and then considering equilibrium of a resulting infinitesimal element (its free body diagram is shown in Figure 2.55) along with taking appropriate limit process, we obtain the discontinuity conditions of the axial force, shear force and bending moment:

$$F_{AR} = F_{AL} \tag{2.40}$$

$$V_{AR} = V_{AL} \tag{2.41}$$

$$M_{AR} = M_{AL} - M_o \tag{2.42}$$

Equations (2.40) and (2.42) imply that at a location where the concentrated moment is applied, the bending moment is discontinuous with the magnitude of the jump equal to the magnitude of the concentrated moment while the axial force and the shear force are still continuous. In particular, the bending moment experiences a positive jump if the concentrated moment is negative (or directs in a clockwise direction or the local z-direction of the member) and it experiences a negative jump if the concentrated moment is positive (or directs in a counter clockwise direction or opposite Z-direction).

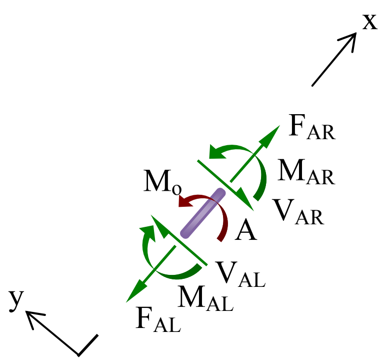


Figure 2.55: FBD of infinitesimal element containing a point where concentrated moment is applied

Finally, we state without proof that points of loading discontinuity such as points where the distributed longitudinal or transverse load is discontinuous and points where both the distributed loads are continuous but change their distribution do not produce the discontinuity in the axial force, the shear force and the bending moment, i.e.

$$F_{AR} = F_{AL} \tag{2.43}$$

$$V_{AR} = V_{AL} \tag{2.44}$$

$$M_{AR} = M_{AL} \tag{2.45}$$

where A denotes a point where the distributed load is discontinuous or change its distribution.

2.6.5.4 Procedures for constructing AFD, SFD and BMD

The differential formula (2.28)-(2.30), the integral formula (2.31)-(2.33) and the discontinuity conditions (2.34)-(2.45) are basic components essential for constructing the AFD, SFD and BMD of a frame. In particular, the three integral formula (2.31)-(2.33) are employed to obtain the axial force, the shear force and the bending moment at the right end of any segment when values of those quantities at the left end are known and there is no point of loading discontinuity within the segment. The three differential formula (2.28)-(2.30) are then used to identify the type of a curve that connects a part of the AFD, SFD and BMD over a segment where values of the axial force, shear force and bending moment are already known at its ends. The discontinuity conditions are used to dictate the jump of the shear force and bending moment in the AFD, SFD and BMD where the concentrated forces and moments are present. Here, we summarize standard procedures or guidelines for constructing the AFD, SFD and BMD of a frame.

- Determine all support reactions
- Identify and mark points of loading discontinuity, e.g. supports, points where concentrated forces and moments are applied, and points where distributed load changes its distribution
- Identify and mark points where members change their orientation
- Separate a given frame into several members using points where members change their orientation
- Identify all possible segments within each member such that points of loading discontinuity must be at the ends of each segment
- Identify a point that the axial force, the shear force and the bending moment are known (in general, a point containing only one member is chosen since all forces and moments at that point are always known once the reactions are already determined.)

- Drawing the SFD as follow: (i) start with a member where the shear force is known at one of its ends, (ii) use the differential formula (2.29), the integral formula (2.32) and the discontinuity conditions (2.35), (2.38), (2.41) and (2.44) to construct the SFD over each segment in the selected member (procedures are similar to those used for the case of beams), (iii) choose the next member where the shear force is known at one end and then follow step (ii), and (iv) repeat step (iii) until all members are considered
- Drawing the BFD as follow: (i) start with a member where the bending moment is known at one of its ends, (ii) use the differential formula (2.30), the integral formula (2.33) and the discontinuity conditions (2.36), (2.39), (2.42) and (2.45) to construct the BMD over each segment in the selected frame member (procedures are similar to those used for the case of beams), (iii) choose the next frame member where the bending moment is known at one end and then follow step (ii), and (iv) repeat step (iii) until all members are considered
- drawing the AFD as follow: (i) start with a member where the axial force is known at one of its ends, (ii) use the differential formula (2.28), the integral formula (2.31) and the discontinuity conditions (2.34), (2.37), (2.40) and (2.43) to construct the AFD over each segment in the selected frame member, (iii) choose the next frame member where the axial force is known at one end and then follow step (ii), and (iv) repeat step (iii) until all members are considered

Once the AFD, SFD and BMD are completed, one can identify both the magnitude and location of the maximum axial force, maximum shear force and maximum bending moment. In general, the maximum axial force and shear force can occur at the supports, the locations where the distributed load q vanishes, the locations where the concentrated forces are applied, and the locations where members change their orientation. Similarly, the maximum bending moment can occur at supports, locations where the shear force vanishes, locations where the shear force changes its sign, locations where the concentrated moments are applied, and locations where members change their orientation.

2.6.6 Sketch of Qualitative Elastic Curve

In this section, we demonstrate how to sketch a qualitative elastic curve or deformed curve of a frame under applied loads once the bending moment diagram is constructed. The key assumptions employed are those associated with Euler-Bernoulli beam theory utilized in the sketch of an elastic curve of beams; i.e. (i) frame is made from a linearly elastic material; (ii) plane section remains plane before and after undergoing deformation; (iii) no shear deformation; and (iv) no axial deformation. According to a kinematics assumption of deformation of the cross section, it is sufficient to represent any frame member by their axis and, therefore, the elastic or deformed curve is the deformed configuration of such axis.

Let u(x), v(x) and θ(x) be the longitudinal component of the displacement, transverse component of the displacement, and the rotation at any point x within a frame member as shown in Figure 2.56. The displacement u(x) and v(x) are considered to be positive if they direct in the positive local x-direction and local y-direction, respectively, and the rotation θ(x) is considered to be positive if it directs in a counter clockwise direction or the local z-direction.

By assuming that the displacement and the rotation are infinitesimally small in comparison with the characteristic length of the frame, the transverse displacement v(x), the rotation θ(x), and the curvature κ(x) are related through the relations

$$\theta(x) = \frac{dv}{dx} \tag{2.46}$$

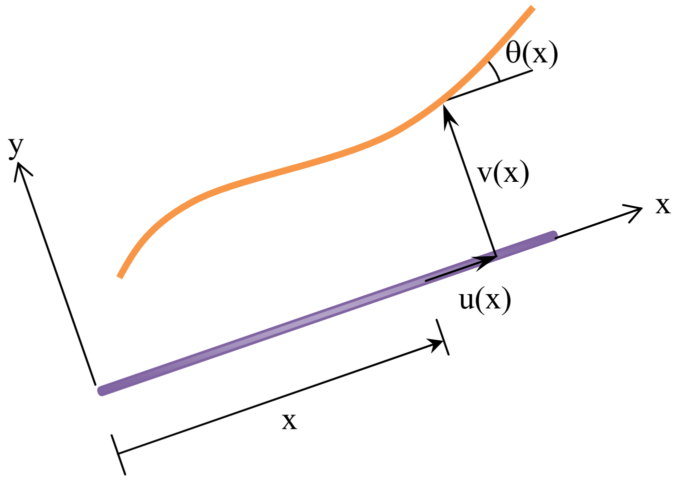


Figure 2.56: Schematics indicating the longitudinal and transverse displacement and the rotation at any point x within a frame member

$$\kappa(x) = \frac{d\theta}{dx} = \frac{d^2v}{dx^2} \tag{2.47}$$

It is evident from the definition (2.47) that the curvature κ(x) is positive if the transverse displacement is concave upward with respect to the local coordinate system {x, y, z}, negative if the displacement is concave downward, and zero if it is an inflection point. In addition, a direct consequence of the small displacement and rotation assumption and the kinematic assumption (iv) leads to that the longitudinal displacement is constant for the entire member or, equivalently, the projection of the deformed curve to the undeformed axis possesses the same length as that of the undeformed member. By considering the deformation of the cross section, employing material constitutive, and computing the moment resultant of the cross section, it leads to a well-known moment-curvature relationship

$$\kappa(x) = \frac{M(x)}{EI} \tag{2.48}$$

where E is Young modulus and I is the moment inertia of the cross section. It can be deduced from the relations (2.47) and (2.48) that

- A segment of a frame possessing the *positive* bending moment undergoes a *positive* curvature and, as a result, leading to a concave *upward* elastic curve;
- A segment of a frame possessing the *negative* bending moment undergoes a *negative* curvature and, as a result, leading to a concave *downward* elastic curve;
- A segment of a frame possessing the *zero* bending moment undergoes a *zero* curvature and, as a result, leading to a straight-line elastic curve; and
- A point within a frame where the bending moment changes sign at that point is an inflection point on the elastic curve.

To sketch the qualitative elastic curve, the following procedures can be used:

- Construct BMD for the entire beam
- Use equation (2.48) to identify the shape of elastic curve at any segment of the frame

- Patch all segments of elastic curve together
- Check compatibility with all supports and internal releases.

It is important to emphasize that the longitudinal displacement at any point is continuous except at the axial release, that the transverse displacement at any point is continuous except at the shear release, and that the rotation at any point is continuous except at the moment release (hinge). Schematics shown in Figure 2.57 indicate the deformation of a segment with no internal release and the discontinuity induced at the axial release, shear release and the moment release.

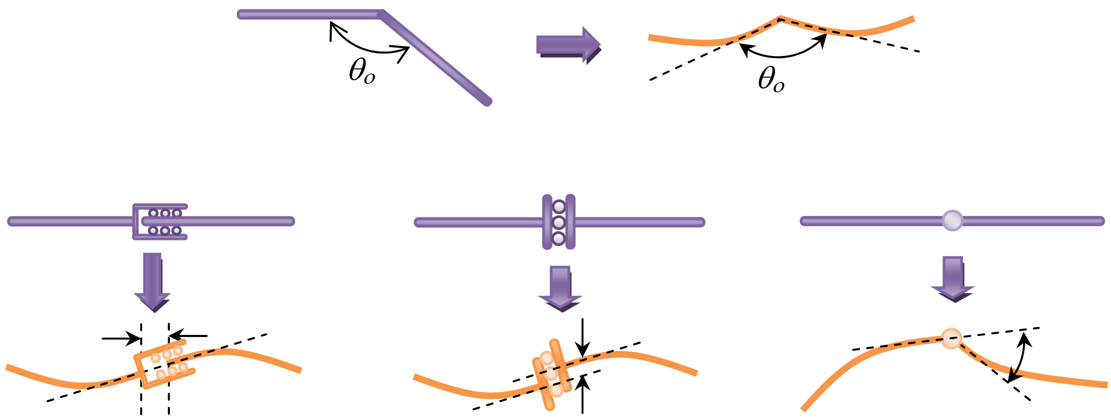
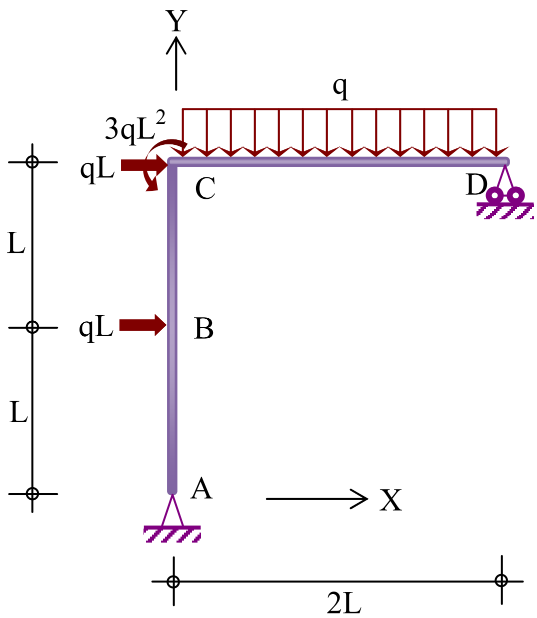
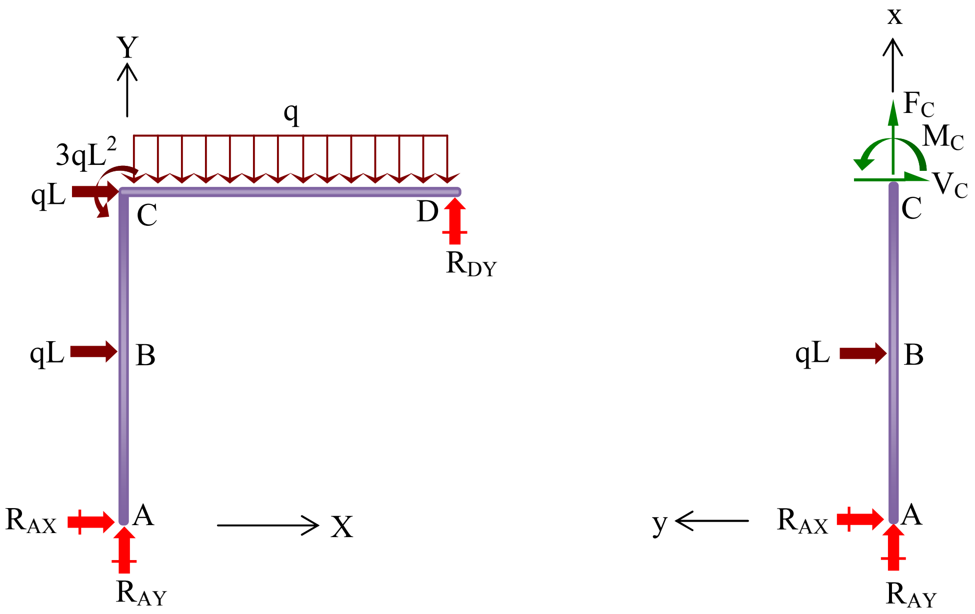


Figure 2.57: Deformation of a segment with no internal release and the discontinuity induced at the axial release, shear release and the moment release

Example 2.15 Determine all support reactions and draw AFD, SFD, BMD and elastic curve of a frame shown below



Solution The given frame is statically determinate (i.e. $r_a = 2 + 1 = 3$, $n_m = 2(3) = 6$, $n_j = 3(3) = 9$, $n_c = 0$, then DI = 3 + 6 – 9 – 0 = 0); thus, all support reactions and the internal forces at any location can be determined from static equilibrium. Since the number of support reactions is equal to 3, they can be obtained from equilibrium of the entire structure. The FBD of the entire structure and details of calculation are shown below:



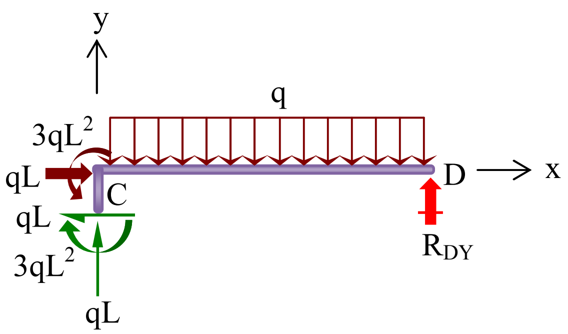
$[\Sigma M_A = 0]$ ↺+	:	$2R_{DY}L - (qL)(L) - (qL)(2L) - (2qL)(L) + 3qL^2 = 0$
		$R_{DY} = qL$ Upward
$[\Sigma F_X = 0]$ →+	:	$R_{AX} + qL + qL = 0$
		$R_{AX} = -2qL$ Leftward
$[\Sigma F_Y = 0]$ ↑+	:	$R_{AY} + R_{DY} - 2qL = 0$
		$R_{AY} = qL$ Upward

For the given frame, there is only one point where members change their orientation, i.e. a point C; therefore, only two frame members, a member AC and a member CD, are considered. For the member AC, there is only one point of loading discontinuity (excluding the two ends of the member), i.e. a point B where a concentrated transverse force is applied, while the member CD contains no point of loading discontinuity. Since all support reactions are already determined, the axial force, shear force and bending moment at the point A are already known. First, let us construct the AFD, SFD and BMD of the member AC. The differential formula (2.28)-(2.30), the integral formula (2.31)-(2.33) and the discontinuity conditions (2.34)-(2.45) are utilized. The local coordinate system for this particular member is shown in above figure.

- AFD
  - No point where the concentrated longitudinal force is applied + no longitudinal distributed load being applied for the entire member ⇒ it is sufficient to consider only one segment AC
  - $F_A = -R_{AY} = -qL$
  - No longitudinal distributed load being applied for the entire member + equation (2.31) ⇒ $F_C = F_A + 0 = -qL$
  - No longitudinal distributed load being applied for the entire member + equation (2.28) ⇒ AFD over the member is a horizontal straight line

- SFD
  - A concentrated transverse force is applied at point B ⇒ member AC is divided into two segments, AB and BC
  - $V_A = -R_{AX} = 2qL$
  - There is no transverse distributed load over the segment AB + equation (2.32) ⇒ $V_{BL} = V_A + 0 = 2qL$
  - There is no transverse distributed load over the segment AB + equation (2.29) ⇒ SFD over the segment AB is a horizontal straight line
  - The negative concentrated transverse force is applied at point B + equation (2.38) ⇒ there is a jump of the shear force at point B ⇒ $V_{BR} = V_{BL} - qL = qL$
  - There is no transverse distributed load over the segment BC + equation (2.32) ⇒ $V_C = V_{BR} + 0 = qL$
  - There is no transverse distributed load over the segment BC + equation (2.29) ⇒ SFD over the segment BC is a horizontal straight line
- BMD
  - No point where the concentrated moment is applied + SFD over the member AC⇒ it is sufficient to consider only two segments, AB and BC
  - $M_A = 0$
  - Area of the SFD over the segment AB is (2qL)(L) + equation (2.33) ⇒ $M_{BL} = M_A + (2qL)(L) = 2qL^2$
  - The shear force is constant and positive over the segment AB + equation (2.30) ⇒ BMD over the segment is a rising straight line
  - There is no concentrated moment applied at point B ⇒ there is no jump of the bending moment at point B ⇒ $M_{BR} = M_{BL} = 2qL^2$
  - Area of the SFD over the segment BC is (qL)(L) + equation (2.33) ⇒ $M_C = M_{BR} + (qL)(L) = 3qL^2$
  - The shear force is constant and positive over the segment BC + equation (2.30) ⇒ BMD over the segment is a rising straight line

Next, let us consider the member CD. Once the axial force, shear force and bending moment at the point C of the member AC are determined, the axial force, shear force and bending moment at the point C of the member CD can readily be obtained. The FBD of the member CD and the corresponding local coordinate system are shown in the figure below.



- AFD
  - No point where the concentrated longitudinal force is applied + no longitudinal distributed load being applied for the entire member ⇒ it is sufficient to consider only one segment CD
  - $F_C = qL - qL = 0$
  - No longitudinal distributed load being applied for the entire member + equation (2.31) ⇒ $F_D = F_C + 0 = 0$ ⇒ consistent with condition at the point D

- No longitudinal distributed load being applied for the entire member + equation (2.28) $\Rightarrow$ AFD over the member is a horizontal straight line

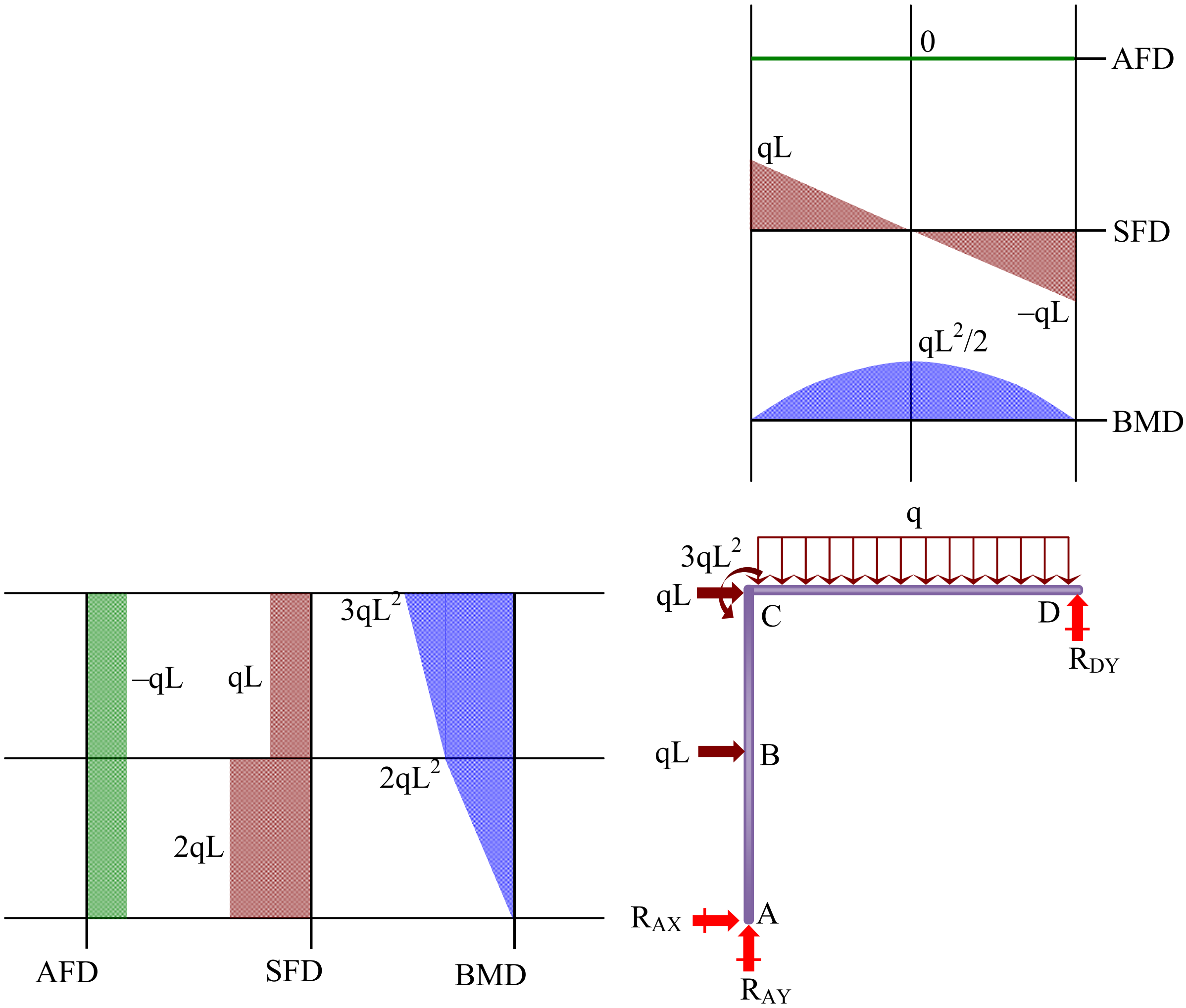
- SFD
  - No point where the concentrated transverse force is applied + the distributed transverse load is continuous for the entire member $\Rightarrow$ it is sufficient to consider only one segment CD
  - $V_C = qL$
  - A negative uniform distributed transverse load is applied over the segment CD + equation (2.32) $\Rightarrow$ $V_D = V_C + (-q)(2L) = -qL$ $\Rightarrow$ consistent with condition at the point D
  - A negative uniform distributed load is applied over the segment CD + equation (2.29) $\Rightarrow$ SFD over the segment AB is a dropping straight line
- BMD
  - No point where the concentrated moment is applied + SFD over the member CD $\Rightarrow$ it is sufficient to consider only two segments CE and ED where E is the mid point of the segment CD
  - $M_C = 3qL^2 - 3qL^2 = 0$
  - Area of the SFD over the segment CE is (qL)(L)/2 + equation (2.33) $\Rightarrow$ $M_{EL} = M_C + (qL)(L)/2 = qL^2/2$
  - The shear force is positive and decreases monotonically in magnitude over a segment CE + equation (2.30) $\Rightarrow$ BMD over the segment is a rising and concave downward curve
  - There is no concentrated moment applied at point E $\Rightarrow$ there is a jump of the bending moment at point E $\Rightarrow$ $M_{ER} = M_{EL} = qL^2/2$
  - Area of the SFD over the segment ED is (–qL)(L)/2 + equation (2.33) $\Rightarrow$ $M_D = M_{ER} + (-qL)(L)/2 = 0$ $\Rightarrow$ consistent with condition at the point D
  - The shear force is negative and increases monotonically in magnitude over a segment ED + equation (2.30) $\Rightarrow$ BMD over the segment is a dropping and concave downward curve

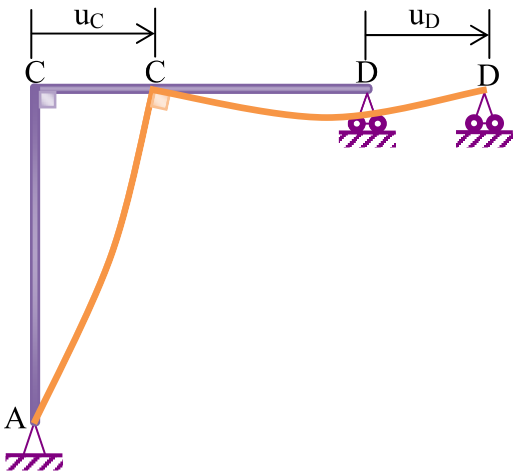
The AFD, SFD, and BMD of the entire frame are shown in the figure below. The maximum axial force, shear force and maximum bending moment and the location where they occur are summarized as follow: for a member AC, the maximum negative axial force is equal to qL occurring at the entire segment AC, the maximum positive shear is equal to 2qL occurring at the entire segment AB, and the maximum positive bending moment is equal to $3qL^2$ occurring at a point C; for the member CD, the maximum positive shear force is equal to qL occurring at a point C, the maximum negative shear force is equal to qL occurring at point D, and the maximum positive bending moment is equal to $qL^2/2$ occurring at a point E.

From movement constraints provided by roller and pinned supports and the BMD shown below, we obtain following information that is useful for sketching an elastic curve:

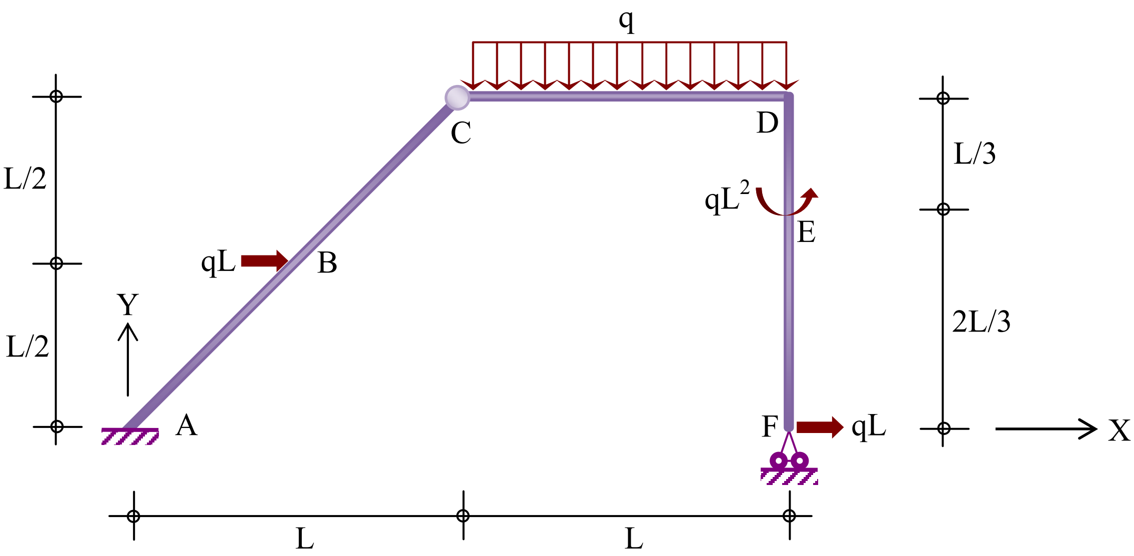
- Point A: pinned support $\Rightarrow$ there is no vertical and horizontal displacements at this point
- Point C: rigid joint $\Rightarrow$ both the displacement and rotation are continuous at this point
- Point D: roller support $\Rightarrow$ there is no vertical displacement at this point while the horizontal displacement and rotation are allowed
- Segment AB: bending moment is positive $\Rightarrow$ the elastic curve of this segment must be concave upward
- Segment BC: bending moment is positive $\Rightarrow$ the elastic curve of this segment must be concave upward
- Segment CD: bending moment is positive $\Rightarrow$ the elastic curve of this segment must be concave upward

- Length constraint of member AC: the vertical displacement at point C must vanish, i.e. $v_C = 0$
- Length constraint of member CD: the horizontal displacement at point C and point D must be identical, i.e. $u_C = u_D$



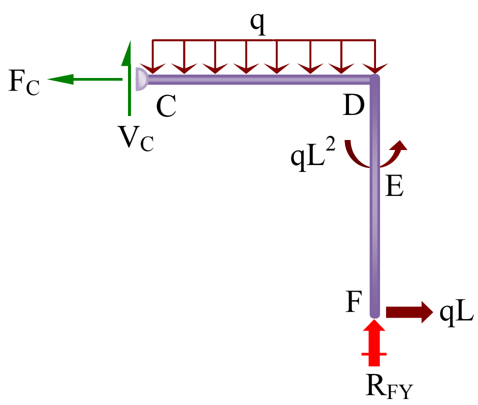


Example 2.16 Determine all support reactions and draw AFD, SFD, BMD and elastic curve of the statically determinate frame structure shown below.

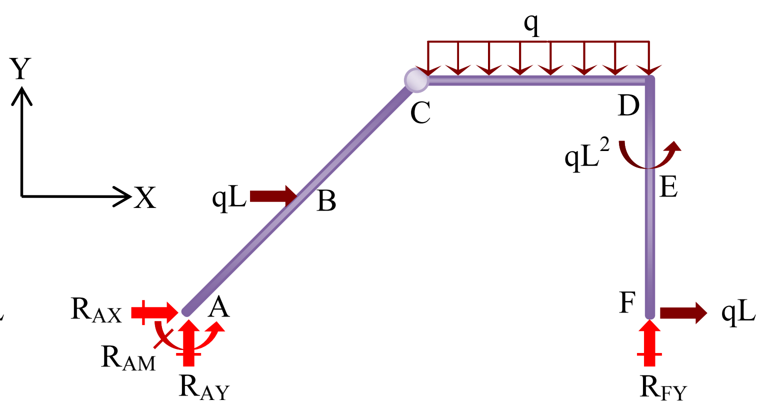


Solution The given frame is statically determinate (i.e. $r_a = 3 + 1 = 4$, $n_m = 3(3) = 9$, $n_j = 4(3) = 12$, $n_c = 1$, then $DI = 4 + 9 - 12 - 1 = 0$); thus, all support reactions and the internal forces at any location can be determined from static equilibrium. However, the number of independent equilibrium equations that can be set up for the entire frame is $n_{et} = 3 < r_a$; thus, the support reactions cannot be obtained from equilibrium of the entire structure. To overcome this problem, an additional equation associated with the presence of a hinge at point C, i.e. M = 0 at C, must be employed.

By introducing a cut at the point C and employing moment equilibrium of the right part of the frame, the reaction R_{FY} can readily be determined and, by considering equilibrium of the entire frame, the rest of reactions can be computed. Details of calculations are shown below:



FBD of portion CDF



FBD of entire frame

Equilibrium of portion CDF

$[\Sigma M_C = 0]$ ↺+ : $R_{FY}L + (qL)(L) + qL^2 - (qL)(L/2) = 0$

$R_{FY} = -3qL/2$ Downward

<u>Equilibrium of entire frame</u>

$[\Sigma F_X = 0]$ $\rightarrow +$: $R_{AX} + qL + qL = 0$

$R_{AX} = -2qL$ Leftward

$[\Sigma M_A = 0]$ ↺+ : $R_{AM} + 2R_{FY}L + qL^2 - (qL)(3L/2) - (qL)(L/2) = 0$

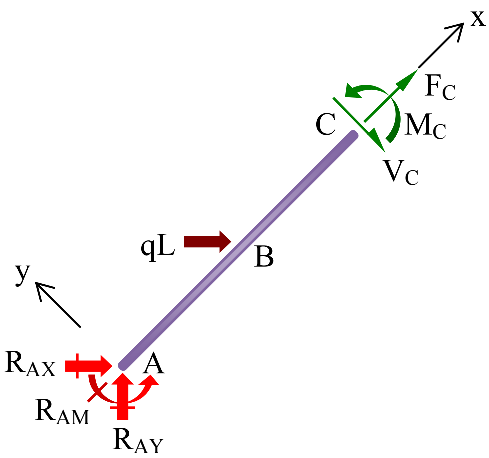
$R_{AM} = 4qL^2$ CCW

$[\Sigma F_Y = 0]$ $\uparrow +$: $R_{AY} + R_{FY} - qL = 0$

$R_{AY} = 5qL/2$ Upward

For the given frame, there are only two points where members change their orientation, i.e. a points C and D; therefore, only three frame members, a member AC, a member CD and a member DF, are considered. In particular, the member AC contains one point of loading discontinuity (i.e. a point B where a concentrated force is applied); the member CD contains no point of loading discontinuity; and the member DF one point of loading discontinuity (i.e. a point E where a concentrated moment is applied). Since all support reactions are already determined, the axial force, shear force and bending moment at the point A are already known.

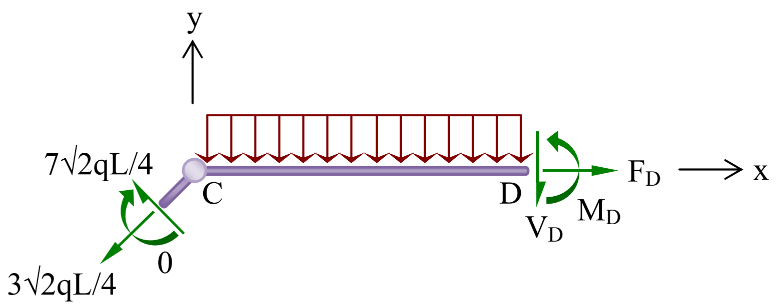
First, let us construct the AFD, SFD and BMD of the member AC. The local coordinate system and the FBD for this particular member are shown in the figure below.



- AFD
  - A concentrated longitudinal force is applied at point B $\Rightarrow$ member AC is divided into two segments, AB and BC
  - $F_A = -R_{AX}\cos 45^o - R_{AY}\sin 45^o = -\sqrt{2}qL/4$
  - No longitudinal distributed load being applied to the segment AB + equation (2.31) $\Rightarrow F_{BL} = F_A + 0 = -\sqrt{2}qL/4$
  - No longitudinal distributed load being applied to the segment AB + equation (2.28) $\Rightarrow$ AFD over the segment AB is a horizontal straight line
  - The positive concentrated longitudinal force is applied at point B + equation (2.34) $\Rightarrow$ there is a jump of the axial force at point B $\Rightarrow F_{BR} = F_{BL} - qL\cos 45^o = -3\sqrt{2}qL/4$
  - No longitudinal distributed load being applied to the segment BC + equation (2.31) $\Rightarrow F_C = F_{BR} + 0 = -3\sqrt{2}qL/4$
  - No longitudinal distributed load being applied to the segment BC + equation (2.28) $\Rightarrow$ AFD over the segment BC is a horizontal straight line

- SFD
  - A concentrated transverse force is applied at point B $\Rightarrow$ member AC is divided into two segments, AB and BC
  - $V_A = -R_{AX}\sin 45^o + R_{AY}\cos 45^o = 9\sqrt{2}qL/4$
  - There is no transverse distributed load over the segment AB + equation (2.32) $\Rightarrow$ $V_{BL} = V_A + 0 = 9\sqrt{2}qL/4$
  - There is no transverse distributed load over the segment AB + equation (2.29) $\Rightarrow$ SFD over the segment AB is a horizontal straight line
  - The negative concentrated transverse force is applied at point B + equation (2.38) $\Rightarrow$ there is a jump of the shear force at point B $\Rightarrow$ $V_{BR} = V_{BL} - qL\sin 45^o = 7\sqrt{2}qL/4$
  - There is no transverse distributed load over the segment BC + equation (2.32) $\Rightarrow$ $V_C = V_{BR} + 0 = 7\sqrt{2}qL/4$
  - There is no transverse distributed load over the segment BC + equation (2.29) $\Rightarrow$ SFD over the segment BC is a horizontal straight line
- BMD
  - No point where the concentrated moment is applied + SFD over the member AC $\Rightarrow$ it is sufficient to consider only two segments, AB and BC
  - $M_A = -R_{AM} = -4qL^2$
  - Area of the SFD over the segment AB is $(9\sqrt{2}qL/4)(\sqrt{2}L/2)$ + equation (2.33) $\Rightarrow$ $M_{BL} = M_A + (9\sqrt{2}qL/4)(\sqrt{2}L/2) = -7qL^2/4$
  - The shear force is constant and positive over the segment AB + equation (2.30) $\Rightarrow$ BMD over the segment is a rising straight line
  - There is no concentrated moment applied at point B $\Rightarrow$ there is no jump of the bending moment at point B $\Rightarrow$ $M_{BR} = M_{BL} = -7qL^2/4$
  - Area of the SFD over the segment BC is $(7\sqrt{2}qL/4)(\sqrt{2}L/2)$ + equation (2.33) $\Rightarrow$ $M_C = M_{BR} + (7\sqrt{2}qL/4)(\sqrt{2}L/2) = 0$
  - The shear force is constant and positive over the segment BC + equation (2.30) $\Rightarrow$ BMD over the segment is a rising straight line

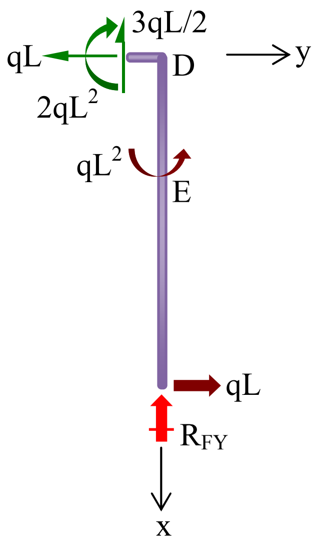
Next, let us construct the AFD, SFD and BMD of the member CD. The local coordinate system and the FBD for this particular member are shown in the figure below.



- AFD
  - No point of loading discontinuity within the member $\Rightarrow$ it is sufficient to consider only one segment
  - $F_C = (7\sqrt{2}qL/4)\cos 45^o - (3\sqrt{2}qL/4)\sin 45^o = qL$
  - No longitudinal distributed load being applied to the member CD + equation (2.31) $\Rightarrow$ $F_D = F_C + 0 = qL$
  - No longitudinal distributed load being applied to the member CD + equation (2.28) $\Rightarrow$ AFD over the segment CD is a horizontal straight line

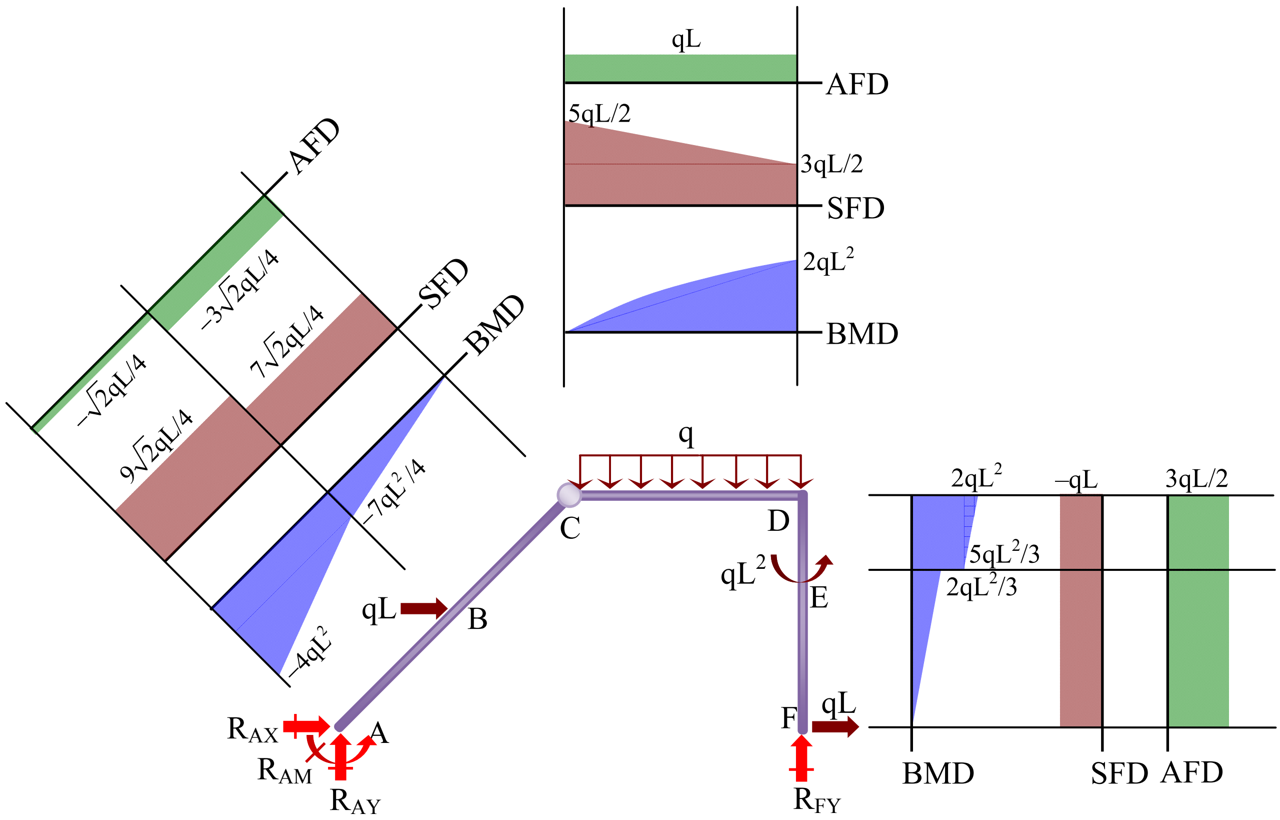
- SFD
  - No point of loading discontinuity within the member ⇒ it is sufficient to consider only one segment
  - $V_C = (7\sqrt{2}qL/4)\sin 45^o + (3\sqrt{2}qL/4)\cos 45^o = 5qL/2$
  - A negative uniform distributed transverse load is applied over the member CD + equation (2.32) ⇒ $V_D = V_C - (q)(L) = 3qL/2$
  - A negative uniform distributed transverse load is applied over the member CD + equation (2.29) ⇒ SFD over the member CD is a dropping straight line
- BMD
  - No point of loading discontinuity within the member + SFD over the member CD⇒ it is sufficient to consider only one segment
  - $M_C = 0$
  - Area of the SFD over the segment AB is (5qL/2+3qL/2)(L/2) + equation (2.33) ⇒ $M_D = M_C + (5qL/2+3qL/2)(L/2) = 2qL^2$
  - The shear force is positive and decreases monotonically in magnitude over the member CD + equation (2.30) ⇒ BMD over the segment is a rising and concave downward curve

Finally, let us construct the AFD, SFD and BMD of the member DF. The local coordinate system and the FBD for this particular member are shown in the figure below.



- AFD
  - No point of longitudinal loading discontinuity within the member ⇒ it is sufficient to consider only one segment
  - $F_D = 3qL/2$
  - No longitudinal distributed load being applied to the member DF + equation (2.31) ⇒ $F_F = F_D + 0 = 3qL/2$ ⇒ consistent with condition at the point F
  - No longitudinal distributed load being applied to the member DF + equation (2.28) ⇒ AFD over the segment DF is a horizontal straight line
- SFD
  - No point of transverse loading discontinuity within the member ⇒ it is sufficient to consider only one segment
  - $V_D = -qL$
  - No transverse distributed load being applied to the member DF + equation (2.32) ⇒ $V_F = V_D + 0 = -qL$ ⇒ consistent with condition at the point F

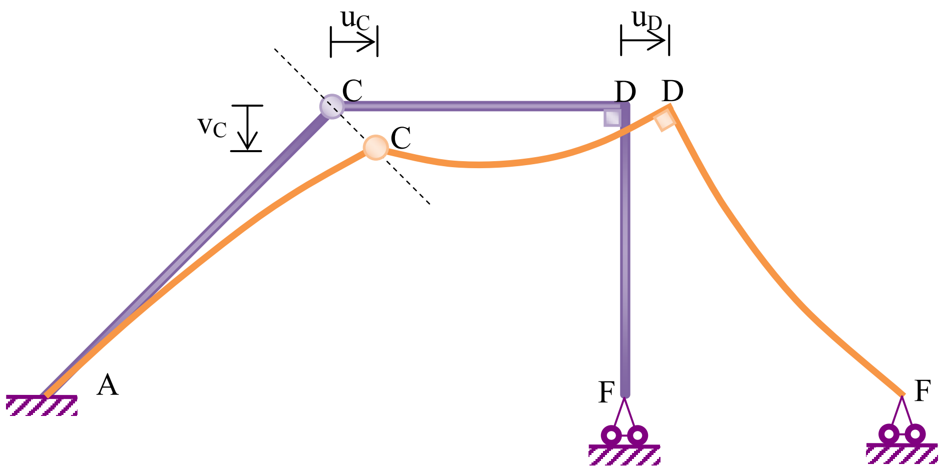
  - No transverse distributed load being applied to the member DF + equation (2.29) ⇒ SFD over the member DF is a horizontal straight line
- BMD
  - A concentrated moment is applied at point E + SFD over the member DF ⇒ the member DF is divided into two segments, DE and EF
  - $M_D = 2qL^2$
  - Area of the SFD over the segment DE is $(–qL)(L/3)$ + equation (2.33) ⇒ $M_{EL} = M_D + (–qL)(L/3) = 5qL^2/3$
  - The shear force is constant and negative over the segment DE + equation (2.30) ⇒ BMD over the segment is a dropping straight line
  - A positive concentrated moment is applied at point E ⇒ there is a jump of the bending moment at point E ⇒ $M_{ER} = M_{EL} – qL^2 = 2qL^2/3$
  - Area of the SFD over the segment EF is $(–qL)(2L/3)$ + equation (2.33) ⇒ $M_F = M_{ER} + (–qL)(2L/3) = 0$ ⇒ consistent with condition at the point F
  - The shear force is constant and negative over the segment EF + equation (2.30) ⇒ BMD over the segment is a dropping straight line



From movement constraints provided by roller and fixed supports, a moment release and the BMD shown below, we obtain following information that is useful for sketching an elastic curve:

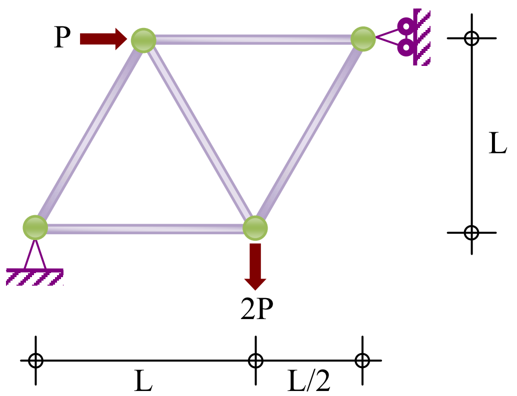
- Point A: fixed support ⇒ there is no vertical and horizontal displacements and rotation at this point
- Point C: hinge joint ⇒ the rotation is discontinuous at this point while the displacement is continuous
- Point D: rigid joint ⇒ both the displacement and rotation are continuous at this point
- Point F: roller support ⇒ there is no vertical displacement at this point while the horizontal displacement and rotation are allowed

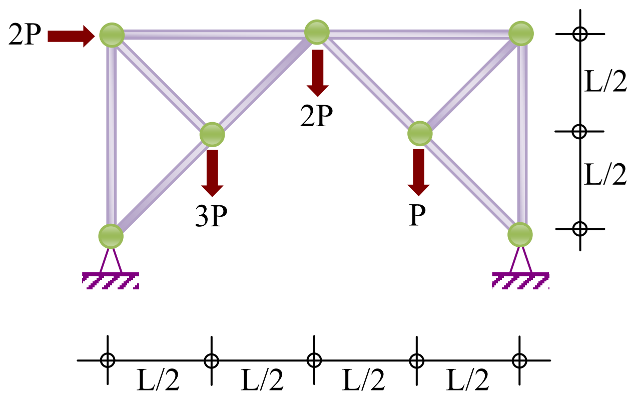
- Segment ABC: bending moment is negative ⇒ the elastic curve of this segment must be concave downward
- Segment CD: bending moment is positive ⇒ the elastic curve of this segment must be concave upward
- Segment DEF: bending moment is positive ⇒ the elastic curve of this segment must be concave upward
- Length constraint of member AC: the vertical displacement and the horizontal displacement at point C must be identical, i.e. $u_C = v_C$
- Length constraint of member CD: the horizontal displacement at point C and point D must be identical, i.e. $u_C = u_D$
- Length constraint of member DF: the vertical displacement at point D must vanish, i.e. $v_D = 0$

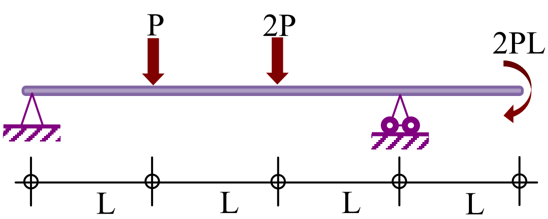


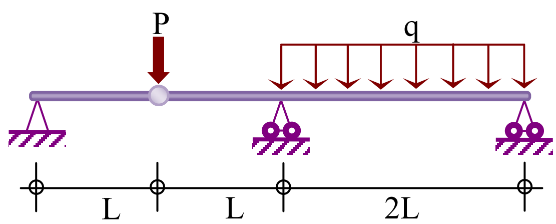
Exercises

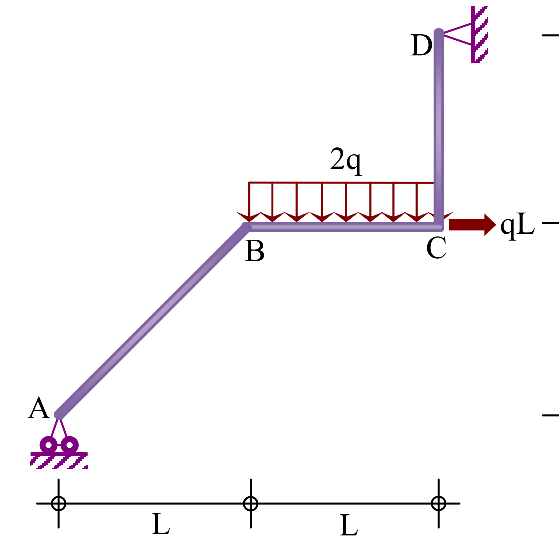
1. Show that structures shown below are externally statically determinate. Sketch free body diagram (FBD) of these structures and then apply static equilibrium equations to determine **all** support reactions.

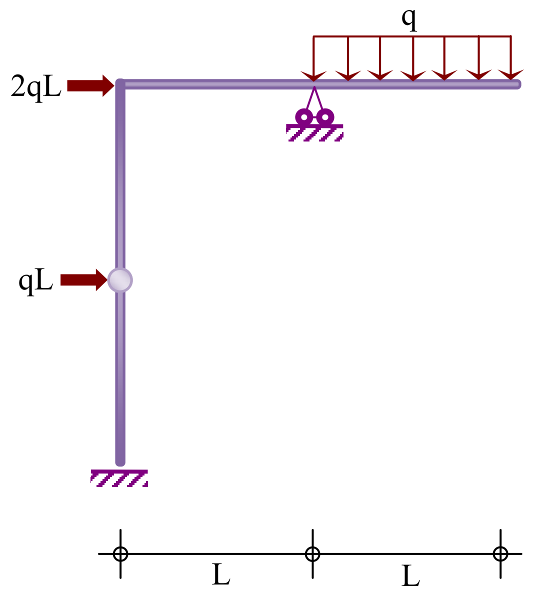




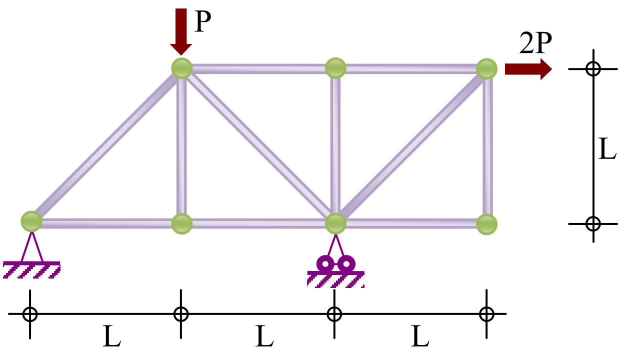


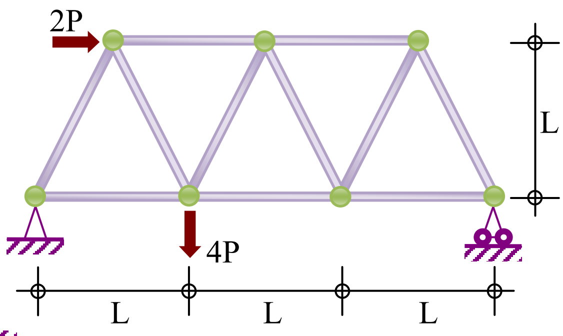


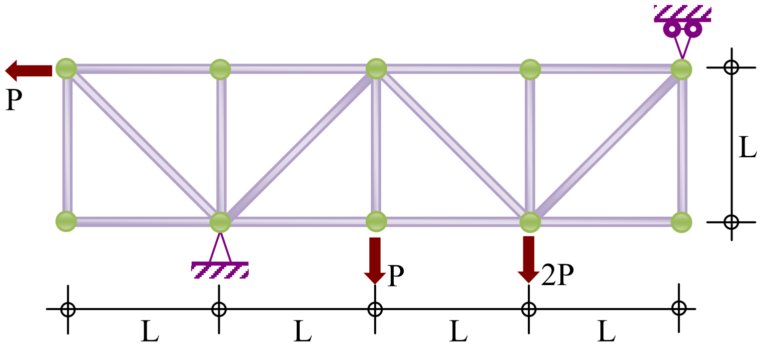


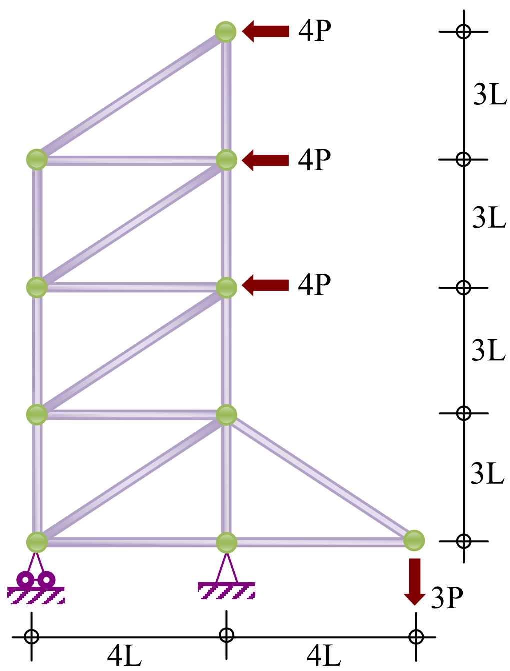


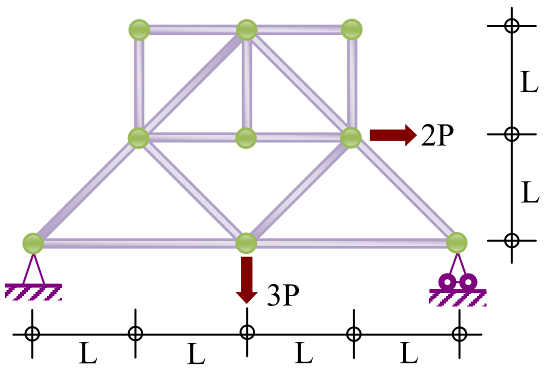
2. For truss structures shown below, you are required to (i) show that they are statically determinate, (ii) sketch FBD and then determine support reactions, (iii) identify zero member forces (if they exist), and (iv) determine the remaining forces using either the method of joints or method of sections.



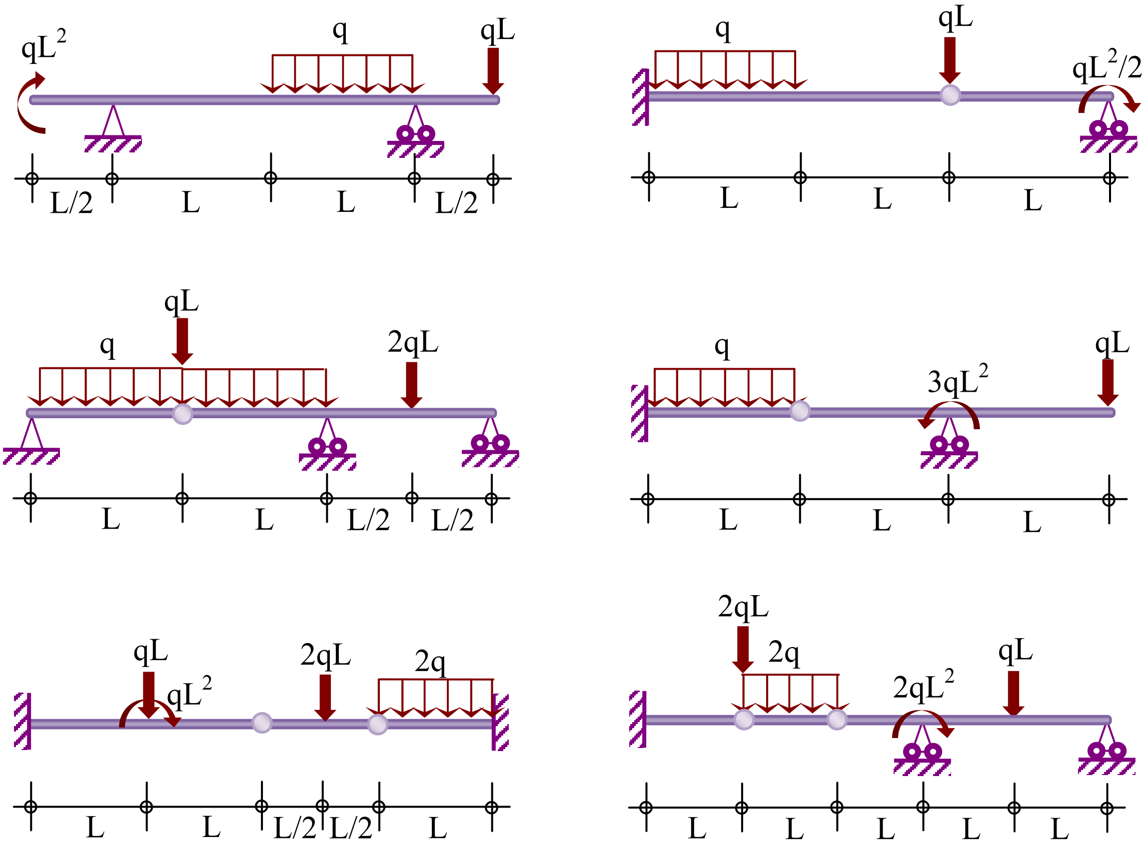




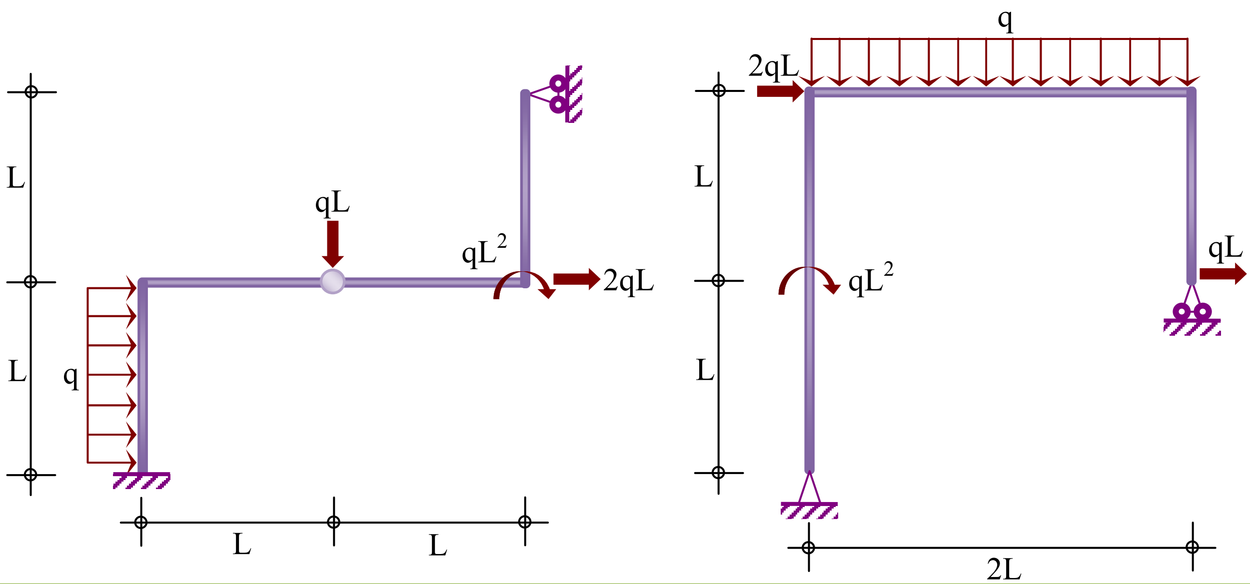


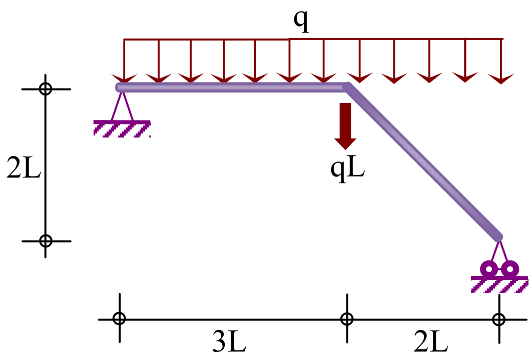


3. For beam structures shown below, you are required to (i) show that they are statically determinate, (ii) sketch FBD and then determine all support reactions, (iii) sketch shear force and bending moment diagrams, (iv) identify the maximum shear force and bending moment, and (v) sketch qualitative elastic curve.



4. For frame structures shown below, you are required to (i) show that they are statically determinate, (ii) sketch FBD and then determine all support reactions, (iii) sketch axial force, shear force and bending moment diagrams, (iv) identify the maximum axial force, shear force and bending moment, and (v) sketch qualitative elastic curve.

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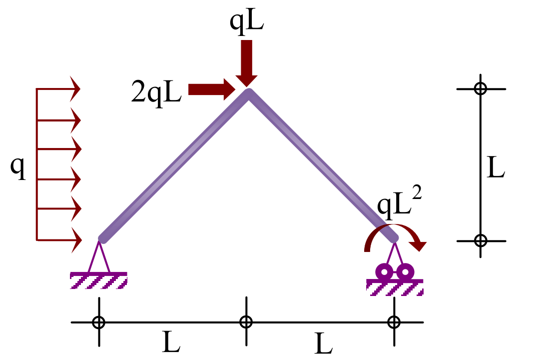
3L
2L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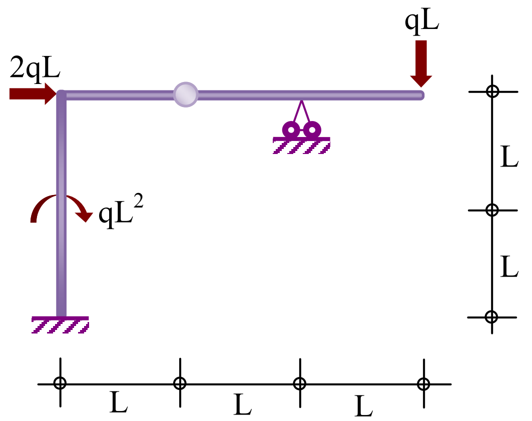
q
qL^2
L
L
L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L
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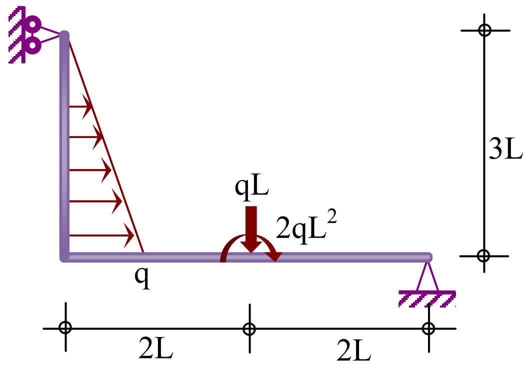
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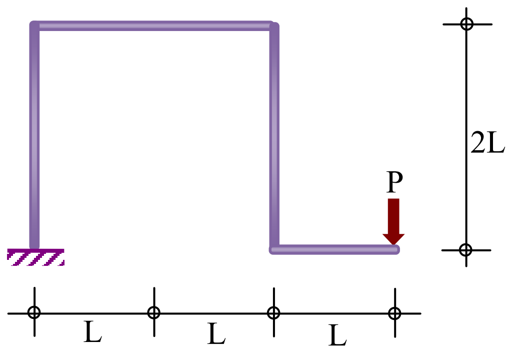
qL
$2qL^2$
q
2L
2L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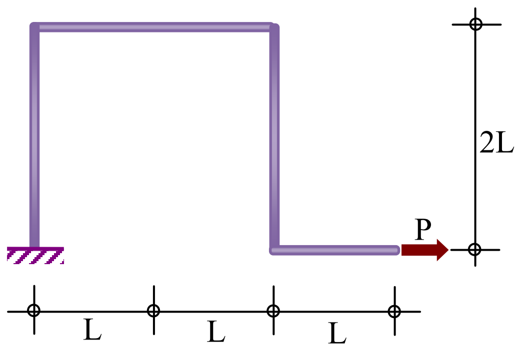
L
L
L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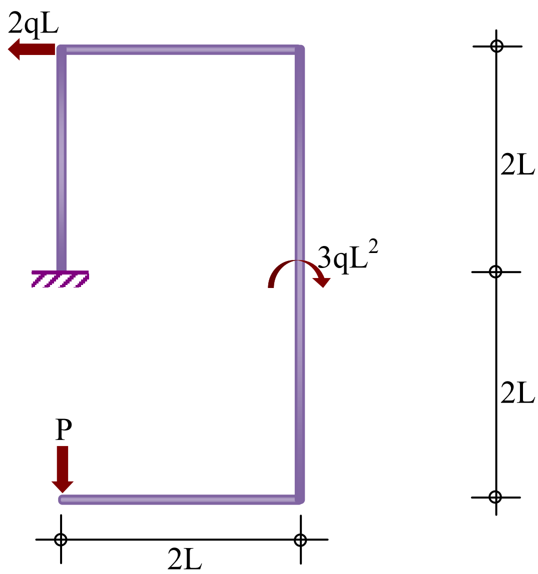
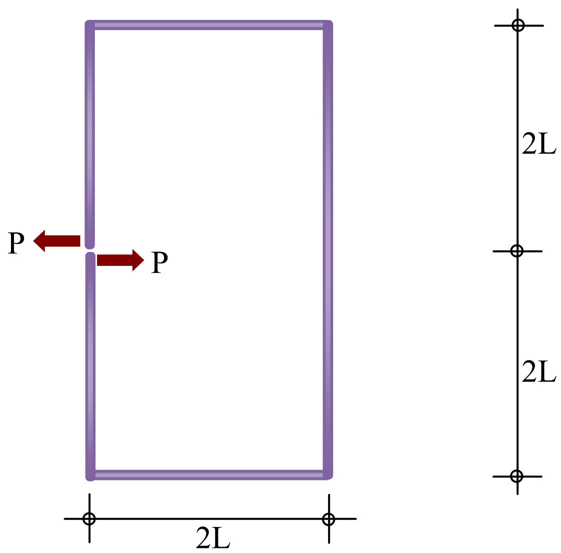

2qL
2L
$3qL^2$
2L
P
2L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P
P
2L
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